

## Twisted brane charges for non-simply connected groups

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**ABSTRACT:** The charges of the twisted branes for strings on the group manifold  $SU(n)/\mathbb{Z}_d$  are determined. To this end we derive explicit (and remarkably simple) formulae for the relevant NIM-rep coefficients. The charge groups of the twisted and untwisted branes are compared and found to agree for the cases we consider.

**KEYWORDS:** Conformal Field Models in String Theory, D-branes.

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## 1. Introduction and background

The charges of D-branes in string theory are believed to be characterised in terms of K-theory [1–3]. For example, for strings that propagate on a group manifold  $G$ , the charge group is conjectured to be the twisted K-theory  ${}^{k+h^\vee}K(G)$  [4, 5], where the twist involves the Wess-Zumino form of the underlying Wess-Zumino-Witten (WZW) model at level  $k$ .

For all simple, simply connected Lie groups  $G$ , the twisted K-theory has been computed in [6] (see also [7, 8]) to be

$${}^{k+h^\vee}K(G) = \underbrace{\mathbb{Z}_{M(\bar{\mathfrak{g}},k)} \oplus \cdots \oplus \mathbb{Z}_{M(\bar{\mathfrak{g}},k)}}_{2^{\text{rk}(\bar{\mathfrak{g}})-1}} \equiv 2^{\text{rk}(\bar{\mathfrak{g}})-1} \cdot \mathbb{Z}_{M(\bar{\mathfrak{g}},k)}, \quad (1.1)$$

where  $M(\bar{\mathfrak{g}}, k)$  is the integer

$$M(\bar{\mathfrak{g}}, k) = \frac{k + h^\vee}{\text{gcd}(k + h^\vee, L)}. \quad (1.2)$$

Here  $h^\vee$  is the dual Coxeter number of the finite dimensional Lie algebra  $\bar{\mathfrak{g}}$  of rank  $\text{rk}(\bar{\mathfrak{g}})$ , and  $L$  only depends on  $\bar{\mathfrak{g}}$  (but not on  $k$ ). In fact, except for the case of  $C_n$  that will not concern us in this paper,  $L$  is

$$L = \text{lcm}\{1, 2, \dots, h - 1\}, \quad (1.3)$$

where  $h$  is the Coxeter number of  $\bar{\mathfrak{g}}$ . For  $\bar{\mathfrak{g}} = A_n$  this formula was derived in [9, 10] (see also [11]), while the formulae in the other cases were checked numerically up to very high levels in [10]. For the classical Lie algebras and  $G_2$  an alternative expression for  $M$  was also found in [8]. If  $G$  is a non-simply connected group manifold much less is known; so far only the case of  $\text{SO}(3) = \text{SU}(2) / \mathbb{Z}_2$  has been worked out in detail [12].

The results of these K-theory analyses should be compared with what can be determined directly in terms of the underlying conformal field theory. The idea behind this approach is that brane configurations that are connected by RG flows should carry the same charge. For the branes  $x \in \mathcal{B}_k^\omega$  of an arbitrary WZW model that preserve the full affine symmetry algebra  $\mathfrak{g}$  up to some automorphism  $\omega$ , this constraint implies in particular, that their charges  $q(x)$  satisfy [9]

$$\dim(\lambda) q(x) = \sum_{y \in \mathcal{B}_k^\omega} \mathcal{N}_{\lambda x}^y q(y). \quad (1.4)$$

Here  $\lambda \in \mathcal{P}_k^+(\bar{\mathfrak{g}})$  is an arbitrary highest-weight representation of the affine Lie algebra  $\mathfrak{g}$  at level  $k$ ,  $\dim(\lambda)$  is the Weyl-dimension of the corresponding representation of the horizontal subalgebra  $\bar{\mathfrak{g}}$ , and  $\mathcal{N}_{\lambda a}^b$  are the NIM-rep coefficients appearing in the Cardy analysis (for an introduction to these matters see [13, 14]).

For the case of the simply connected group manifold, the corresponding WZW model is the charge-conjugation modular invariant; by analogy to the ADE classification for the case of  $\text{SU}(2)$  we shall in the following refer to it as the  $\mathcal{A}$ -modular invariant. The untwisted branes (that correspond to the trivial automorphism  $\omega = \text{id}$ ) can be labelled by integrable highest weights of  $\mathfrak{g}$ ,  $\mathcal{B}_k^{\text{id}} \cong \mathcal{P}_k^+(\bar{\mathfrak{g}})$ , and the NIM-rep  $\mathcal{N}(\mathcal{A})$  agrees with the fusion rules. In this case, the constraints (1.4) were evaluated in [9, 10]. The charges are given (up to rescalings) by the Weyl-dimensions of the corresponding representations,  $q(\lambda) = \dim(\lambda)$ , and the charge is conserved only modulo  $M(\bar{\mathfrak{g}}, k)$ . Thus, the untwisted branes account for one summand  $\mathbb{Z}_{M(\bar{\mathfrak{g}},k)}$  of the K-group (1.1).

For non-trivial outer automorphisms, a similar analysis was carried through in [15–17]. Here, the D-branes are parametrised by  $\omega$ -twisted highest weight representations  $a$  of  $\mathfrak{g}_k$  [18–20], and the NIM-rep coefficients are given by twisted fusion rules [20]. In fact, the NIM-rep is the same as the one describing the untwisted branes of the  $\mathcal{A}^\omega$ -modular invariant that is obtained from the charge conjugation  $\mathcal{A}$ -modular invariant by applying the automorphism  $\omega$  to the left-movers say; for the case of  $SU(n)$  that shall concern us in the following,  $\omega$  is charge conjugation and we shall thus denote the corresponding NIM-rep by  $\mathcal{N}(\mathcal{A}^*)$ .

The twisted representations can be identified with representations of the invariant subalgebra  $\bar{\mathfrak{g}}^\omega$  consisting of  $\omega$ -invariant elements of  $\bar{\mathfrak{g}}$ , and we can view  $\mathcal{B}_k^\omega$  as a subset of  $\mathcal{P}_{k'}^+(\bar{\mathfrak{g}}^\omega)$ , where  $k' = k + h^\vee(\bar{\mathfrak{g}}) - h^\vee(\bar{\mathfrak{g}}^\omega)$ . It was found in [15] that the charge  $q(a)$  of  $a \in \mathcal{B}_k^\omega$  is again (up to rescalings) given by the Weyl dimension<sup>1</sup> of the representation of  $\bar{\mathfrak{g}}^\omega$ ,  $q(a) = \dim(a)$ , and that the charge identities are only satisfied modulo  $M(\bar{\mathfrak{g}}, k)$ . Thus each such class of twisted D-branes accounts for another summand  $\mathbb{Z}_{M(\bar{\mathfrak{g}}, k)}$  of the charge group. Since the number of automorphisms does not grow with the rank, these constructions do not in general account for all the charges of (1.1); for the case of the  $A_n$  series, a proposal for the D-branes that may carry the remaining charges was made in [22, 23] (see also [11]).

In this paper we shall study the D-branes charges for string theory on the non-simply connected group manifolds  $SU(n)/\mathbb{Z}_d$ , where  $d$  is a factor of  $n$ . The corresponding modular invariants are known [24, 25]. By analogy to the  $SU(2)$  case we shall call them the  $\mathcal{D}$ -modular invariants. For quotient groups of  $SU(n)$  there are then two classes of branes that preserve the full affine symmetry up to an automorphism. First there are the untwisted branes for which the automorphism is trivial; the corresponding NIM-rep will be denoted by  $\mathcal{N}(\mathcal{D})$ . Some aspects of this NIM-rep were already studied in [26] where also the charge group was partially determined. As was observed there, the analysis depends crucially on whether  $n(n+1)/d$  is even or odd. In particular, it was found that if  $n(n+1)/d$  is odd, the charge group is surprisingly small; this was called the *pathological case* in [26]. This pathological behaviour may be related to the fact that in these cases there is a second modular invariant that one can consider (in which the fermionic degrees are treated differently); at least for the case of  $SO(3)$  the charge group of this second theory has again the expected size [27].

In this paper we shall only consider the non-pathological case, *i.e.* we shall assume that  $n(n+1)/d$  is even. For this class of theories we shall be able to give a complete description of the NIM-rep  $\mathcal{N}(\mathcal{D})$ . This will also allow us to determine the  $\mathcal{D}$  charge group in more detail; this will be described in section 4.

The main new results of this paper however concern the analysis of the twisted D-branes, *i.e.* the branes that preserve the affine symmetry up to the outer automorphism that corresponds to charge conjugation. Since these branes are naturally in one-to-one correspondence with the untwisted branes of the modular invariant  $\mathcal{D}^*$  (that can be obtained from  $\mathcal{D}$  by charge conjugation) we shall denote the relevant NIM-rep by  $\mathcal{N}(\mathcal{D}^*)$ . As we

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<sup>1</sup>A related proposal was made in [21] based on an analysis for large level.

shall see, we can give a remarkably simple formula for this NIM-rep in all non-pathological cases. In many cases we can furthermore determine the resulting charge groups in detail. This will be explained in sections 2 and 3.

For the case of the simply connected groups it was shown in [15–17] that the charge groups of the untwisted ( $\mathcal{A}$ ) and twisted ( $\mathcal{A}^*$ ) branes coincide. One may wonder whether this is true in general. For the non-pathological cases we study in this paper this also seems to be the case: whenever we can determine both the  $\mathcal{D}$ - and  $\mathcal{D}^*$ -charge groups explicitly, they agree. This agreement is fairly non-trivial since the calculations that are involved for the two cases are at least superficially very different. One may therefore expect that there is a more conceptual explanation of this correspondence; some ideas in this direction are described in section 5. We also note there that this agreement only seems to hold in the non-pathological case; we have found an explicit counterexample, namely  $SU(4)/\mathbb{Z}_4$  at level  $k = 4$  (which is pathological), for which the two charge groups disagree.

The paper is organised as follows. In the following subsection we introduce some more notation. We construct the  $\mathcal{D}^*$  NIM-rep and determine its charge group for the case when  $n$  is odd (which is always non-pathological) in section 2. The non-pathological cases with  $n$  even are dealt with in section 3. In section 4 we give a more detailed description of the  $\mathcal{D}$  NIM-rep in the non-pathological cases and calculate the charge group explicitly (at least for certain classes of examples). A possible relation between the charge groups of  $\mathcal{D}$  and  $\mathcal{D}^*$  is explained in section 5, and section 6 contains some conclusions. We have included a number of appendices in which some technical proofs are given.

### 1.1 Some background material and notation

Before we can begin with the detailed discussion we need to introduce some notation. Recall that the integrable highest weights  $\lambda \in P_+^k(\mathfrak{su}(n))$  consist of all  $n$ -tuples  $\lambda = (\lambda_0; \lambda_1, \dots, \lambda_{n-1})$  of non-negative integers  $\lambda_i$ , where  $\sum_{j=0}^{n-1} \lambda_j = k$ . We will often drop the redundant 0'th Dynkin label  $\lambda_0$ . We shall also often write  $\widehat{\mathfrak{su}}(n)_k$  for ‘affine  $\mathfrak{su}(n)$  at level  $k$ ’. Charge-conjugation  $C$  and the generator  $J$  of simple currents is given by

$$\lambda^* = C\lambda = (\lambda_0; \lambda_{n-1}, \lambda_{n-2}, \dots, \lambda_1) \tag{1.5}$$

$$J\lambda = (\lambda_{n-1}; \lambda_0, \lambda_1, \dots, \lambda_{n-2}) . \tag{1.6}$$

Their effect on the  $S$ -matrix elements is  $S_{C\lambda, \mu} = \overline{S_{\lambda, \mu}}$  and

$$S_{J^j \lambda, \mu} = \exp[2\pi i j t(\mu)/n] S_{\lambda, \mu} , \tag{1.7}$$

where  $t(\mu) = \sum_{j=1}^{n-1} j\mu_j$  is the  $n$ -ality of  $\mu$ . Note that  $C(J^j \lambda) = J^{-j} C\lambda$ , and  $t(C\lambda) = -t(\lambda) \pmod n$ .

Simple currents give rise to symmetries and gradings of fusion rule coefficients

$$N_{J^i \lambda, J^j \mu}^{J^{i+j} \nu} = N_{\lambda \mu}{}^\nu \tag{1.8}$$

$$N_{\lambda \mu}{}^\nu \neq 0 \implies t(\lambda) + t(\mu) = t(\nu) \pmod n . \tag{1.9}$$

Strings moving on the simply connected covering group  $SU(n)$  [24, 25] are described by the WZW model with modular invariant

$$M_{\lambda\mu} = \delta_{\lambda\mu} . \tag{1.10}$$

This modular invariant is sometimes referred to as the  $\mathcal{A}$ -modular invariant.

The centre of  $SU(n)$  is  $\mathbb{Z}_n$ . For any factor  $d$  of  $n$ , we therefore have a quotient group  $SU(n)/\mathbb{Z}_d$ . The corresponding [24, 25] modular invariant is [28]

$$M[d']_{\lambda\mu} = \sum_{j=1}^d \delta_d \left( t(\lambda) + \frac{d' j k'}{2} \right) \delta^{\mu J^{j d'} \lambda} , \tag{1.11}$$

where  $d' = n/d$ , and  $k' = k + n$  if  $k$  and  $n$  are odd, and  $k' = k$  otherwise. In order for this to define a modular invariant partition function we need to have that  $n(n+1)k/d$  is even. If this is the case, we call it the  $\mathcal{D}$ -modular invariant.

The simple current  $J^{d'}$  that will play an important role in the following, has order  $d$ . We call  $\varphi \in P_+^k(\mathfrak{su}(n))$  a *fixed point of order  $m$*  (with respect to  $J^{d'}$ ) if the  $J^{d'}$ -orbit  $\{J^{j d'} \varphi\}$  has cardinality  $d/m$  – in other words,  $m$  divides  $d$ , and  $d/m$  is the smallest positive integer for which  $J^{d'(d/m)} \varphi = J^{n/m} \varphi = \varphi$ . Write  $o(\varphi)$  for the order  $m$  of  $\varphi$ . Note that any solution  $\varphi \in P_+^k(\mathfrak{su}(n))$  to  $J^{n/m} \varphi = \varphi$  looks like  $\varphi = (\bar{\varphi}, \dots, \bar{\varphi})$  ( $m$  copies of  $\bar{\varphi}$ ), where  $\bar{\varphi} = (\varphi_0; \dots, \varphi_{n/m-1}) \in P_+^{k/m}(\mathfrak{su}(n/m))$ . For a given  $n, k, d$ ,  $J^{d'}$  will have fixed points of order  $m$ , when and only when  $m$  divides  $\text{gcd}(d, k)$ . If a fixed point  $\varphi$  has order  $m$ , and  $\Delta$  divides  $m$ , then by  $\varphi^\Delta$  we mean the  $\widehat{\mathfrak{su}}(n/\Delta)_k/\Delta$  weight obtained by retaining only the first  $n/\Delta$  components of  $\varphi$ .

A NIM-rep  $\mathcal{N}$  for  $\widehat{\mathfrak{su}}(n)_k$  can be uniquely specified in two different ways: either by giving the matrices  $\mathcal{N}_{\Lambda_m} = (\mathcal{N}_{\Lambda_m} x^y)$  for all fundamental weights  $\Lambda_m = (0, \dots, 1, \dots, 0)$ ; or by specifying the matrix  $\psi = (\psi_{x\mu})$  which simultaneously diagonalises all matrices  $\mathcal{N}_\lambda = (\mathcal{N}_{\lambda x}^y)$  in the sense that

$$\mathcal{N}_{\lambda x}^y = \sum_{\mu} \psi_{x\mu} \frac{S_{\lambda\mu}}{S_{0\mu}} \psi_{y\mu}^* . \tag{1.12}$$

Here,  $\lambda \in P_+^k(\mathfrak{su}(n))$ ,  $x, y$  are boundary labels, and  $\mu$  are the exponents for the corresponding NIM-rep, *i.e.* the weights that label the possible Ishibashi states (with multiplicity). Alternatively, these are just the exponents of the corresponding modular invariant  $\mathcal{Z}$ , *i.e.* the weights appearing with multiplicity  $\mathcal{Z}_{\mu\mu}$ .

Given a NIM-rep  $\mathcal{N}$ , we can determine the corresponding charge group following [9]. In particular, a *charge assignment* consists of integers  $q(x)$  (one for each boundary label) and  $M$  such that (1.4) is satisfied mod  $M$ . For  $\widehat{\mathfrak{su}}(n)_k$  we have  $h^\vee = h = n$ , and thus

$$M \equiv M(\mathfrak{su}(n), k) = \frac{n+k}{\text{gcd}(n+k, \text{lcm}(1, 2, \dots, n-1))} . \tag{1.13}$$

Furthermore, one knows on general grounds that any charge assignment is always proportional to one defined modulo  $M$ . The sum  $q(x) \equiv q'(x) + q''(x)$  of arbitrary charge

assignments  $q'(x), q''(x)$  (defined mod  $M$ ) is likewise one defined mod  $M$ . The charge assignments thus form an additive group called the *charge group*; for  $\widehat{\text{su}}(n)_k$  it takes the form

$$\mathbb{Z}_{M_1} \oplus \cdots \oplus \mathbb{Z}_{M_t}, \tag{1.14}$$

where  $\mathbb{Z}_m$  denotes the additive group  $\mathbb{Z}/m\mathbb{Z}$ , and where each  $M_i$  divides  $M$ . The decomposition (1.14) becomes unique if in addition we require each  $M_i$  to divide  $M_{i-1}$ . Both  $\mathcal{A}$  and  $\mathcal{A}^*$  have  $t = 1$ , but both  $\mathcal{D}$  and  $\mathcal{D}^*$  can have  $t > 1$ .

After these preliminary remarks we can now study the  $\mathcal{D}^*$  NIM-rep and its associated charge group.

## 2. The $\mathcal{D}^*$ case with $n$ odd

We begin by analysing the charges for  $\mathcal{D}^*$  for  $n$  odd. In section 2.1 we construct the relevant NIM-rep explicitly. In section 2.2 we then determine its charge group. Given the simplicity of our formula for the NIM-rep, we are able to do this in closed form.

### 2.1 The NIM-rep

As before we write  $d' = n/d$ . Since  $n$  is odd, we may write  $n = 2m + 1$ . The NIM-rep for  $\mathcal{A}^*$  on  $\widehat{\text{su}}(n)_k$  was given in section 4.1 of [20]. Recall that the exponents of the charge-conjugate ( $\mathcal{A}^*$ ) modular invariant for  $\text{su}(n)$  consist of all  $\mu \in P_+^k(\text{su}(n))$  with  $C\mu = \mu$ , all with multiplicity 1. The boundary states  $\widehat{a}$  are parametrised by the level  $k$  integrable highest weights of the twisted affine algebra  $A_{n-1}^{(2)}$ , *i.e.* all  $(m+1)$ -tuples  $(a_0; a_1, \dots, a_m)$  of non-negative integers where  $k = a_0 + 2a_1 + 2a_2 + \cdots + 2a_m$ . The  $\psi$ -matrix, diagonalising the NIM-rep, is the modular  $\widehat{S}$ -matrix of  $A_{n-1}^{(2)}$  (see eq. (4.2) in [20]). The NIM-rep coefficients are  $C$ -twisted fusion coefficients of  $A_{n-1, k}$ , and can be expressed in terms of ordinary fusions of both  $B_m^{(1)}$  level  $k+2$  and  $C_m^{(1)}$  level  $(k-1)/2$  [20, 15]. We will write this NIM-rep in the form

$$\lambda.\widehat{a} = \sum_{\widehat{b}} \mathcal{N}(\mathcal{A}^*)_{\lambda\widehat{a}}^{\widehat{b}} \widehat{b}. \tag{2.1}$$

For the  $\mathcal{D}^*$  modular invariant (*i.e.* the charge conjugate of (1.11)) one finds with a little work that the exponents form the multi-set  $\bigcup_{j=1}^d \bigcup_{\mu=C\mu} J^{jd'} \mu$ . That is,  $\mu \in P_+^k(\text{su}(n))$  will have multiplicity 0 unless  $C(J^{jd'} \mu) = J^{jd'} \mu$  for some  $j$ , in which case its multiplicity will equal its order as a  $J^{d'}$ -fixed point. We should stress that this simple result assumes that  $n$  is odd.

We can thus unambiguously specify the exponents of  $M[d']^*$  by pairs  $(\mu, i)$ , where  $\mu = C\mu$  and  $1 \leq i \leq d$ . Likewise, the boundary states for  $M[d']^*$  are given by all pairs  $(\widehat{a}, j)$ ,  $0 \leq j < d$ . The formula for the  $\mathcal{D}^*$  NIM-rep for any  $n, d$  (provided only that  $n$  is odd), is given by the remarkably simple formula

$$\mathcal{D}^* : \quad \boxed{\lambda.(\widehat{a}, j) = (\lambda.\widehat{a}, j + t(\lambda))} \tag{2.2}$$

where the first product is defined by (2.1) and  $j + t(\lambda)$  is to be taken mod  $d$ . The  $\psi$ -matrix for  $M[d']^*$  is

$$\psi(\mathcal{D}^*)_{(\widehat{a},j),(\mu,i)} = \frac{1}{\sqrt{d}} e^{2\pi i ij/d} \psi(\mathcal{A}^*)_{\widehat{a},\mu} . \quad (2.3)$$

The above construction is actually a special case of a much more general one. Let  $\mathcal{N}_{\lambda x^y}$  be any NIM-rep, and let  $J$  be any simple current of the underlying fusion ring, say of order  $m$ . Then there will be a corresponding function [28]  $Q_J : P_+ \rightarrow \frac{1}{m}\mathbb{Z}$  such that

$$S_{J\lambda,\mu} = \exp[2\pi i Q_J(\mu)] S_{\lambda,\mu} . \quad (2.4)$$

Then consider the  $m$ -fold cover where the boundary states are labelled by  $(x, j)$  (where  $j \in \mathbb{Z}_m$ ), and the NIM-rep is defined by  $\lambda.(x, j) = (\lambda.x, j + mQ_J(\lambda))$ . This is really just the tensor product  $\mathcal{N} \otimes \mathbb{Z}_m$  of two NIM-reps. If  $\mu \in \mathcal{E}$  are the exponents of the original NIM-rep  $\mathcal{N}$ , then the exponents of the new NIM-rep will be the multi-set  $\cup_{i=0}^{m-1} J^i \mathcal{E}$ . Now, we know that a NIM-rep is decomposable (into a direct sum of smaller NIM-reps) if and only if the multiplicity of the vacuum 0 as an exponent is larger than one (see [29]). Thus our new NIM-rep  $\mathcal{N} \otimes \mathbb{Z}_m$  will be indecomposable, iff the old NIM-rep  $\mathcal{N}$  was indecomposable and if no non-trivial power of the simple current  $J$  is itself an exponent of  $\mathcal{N}$ .

By these considerations, we know that our proposed NIM-rep  $\mathcal{N}(\mathcal{A}^*) \otimes \mathbb{Z}_d$  has the correct exponents. Furthermore, it is indecomposable since no non-trivial power of  $J^{d'}$  is an exponent of  $\mathcal{N}(\mathcal{A}^*)$ . Thus it follows that the above construction can be taken to define the NIM-rep  $\mathcal{N}(\mathcal{D}^*)$ .

## 2.2 Charges and charge groups

Given that we have such a simple explicit formula for the NIM-rep, we can analyse the corresponding charges in detail. These are integers  $q(\widehat{a}, i)$  satisfying

$$\dim(\lambda) q(\widehat{a}, i) = \sum_{\widehat{b}} \mathcal{N}(\mathcal{A}^*)_{\lambda \widehat{a} \widehat{b}} q(\widehat{b}, i + t(\lambda)) \pmod{M} \quad (2.5)$$

for  $M$  in (1.13). The simplest solution to (2.5) arises if all  $q(\widehat{0}, i)$  are equal (for all  $0 \leq i < d$ ). Then by an easy induction argument from (3.4), (3.9) of [15], we get that *any*  $q(\widehat{a}, i)$  is independent of  $i$ , and equals a solution  $q_{\mathcal{A}^*}(\widehat{a})$  to the charge equations for  $\mathcal{A}^*$ , and so by uniqueness (section 6.2 of [15])  $q(\widehat{a}, i)$  equals the common value of  $q(\widehat{0}, i)$ , times the dimension  $\dim_C(a)$  of the  $C_m$ -representation with highest weight  $a = (a_1, \dots, a_m)$  (see (3.1) of [15]). For  $n$  odd, this uniqueness argument depends on a (safe) conjecture (see the discussion in section 1.2 of [15]).

Thus it follows that  $\mathcal{D}^*$  inherits a charge assignment from  $\mathcal{A}^*$ , namely  $q(\widehat{a}, i) = \dim_C(a)$ . In particular, the charge group for  $\mathcal{D}^*$  contains therefore a summand  $\mathbb{Z}_M$ . By analogy with the  $\mathcal{D}$  charge analysis of [26], we should expect other solutions  $q(\widehat{a}, i)$  to (2.5). However, any of these are uniquely determined by the ‘initial values’  $q(\widehat{0}, i)$ . The reason is that if two solutions  $q(\widehat{a}, i)$ ,  $q'(\widehat{a}, i)$  agree at  $(\widehat{0}, i)$ , then by the argument of the previous paragraph their difference must equal 0 everywhere.

In order to analyse the full charge group we therefore have to study the solutions that are determined by all possible sets of ‘initial values’. One constraint on these can be easily



found as follows. All exponents  $\mu$  of the  $\mathcal{A}^*$  NIM-rep obey  $\mu = \mu^*$ . This implies that the matrices  $\mathcal{N}(\mathcal{A}^*)_\lambda$  and  $\mathcal{N}(\mathcal{A}^*)_{\lambda^*}$  must be equal (they have the same eigenvectors, namely the columns of  $\widehat{S}$ , and the same eigenvalues, namely  $\frac{S_{\lambda^*\mu}}{S_{0\mu}} = \frac{S_{\lambda\mu^*}}{S_{0\mu}} = \frac{S_{\lambda\mu}}{S_{0\mu}}$ ). Also,  $t(\lambda^*) = -t(\lambda) \pmod{n}$  and  $\dim(\lambda^*) = \dim(\lambda)$ . Hence replacing  $\lambda$  with  $\lambda^*$  in (2.5) yields

$$\dim(\lambda) q(\widehat{a}, i) = \sum_{\widehat{b}} \mathcal{N}(\mathcal{A}^*)_{\lambda\widehat{a}} \widehat{b} q(\widehat{b}, i - t(\lambda)) \pmod{M}, \tag{2.6}$$

and comparing this with (2.5) gives our basic constraint

$$\dim(\lambda) q(\widehat{0}, i) = \dim(\lambda) q(\widehat{0}, i + 2t(\lambda)) \pmod{M}, \tag{2.7}$$

valid for any  $\lambda \in P_+^k(\mathfrak{su}(n))$  and  $0 \leq i < d$ . Of course, (2.7) holds for any  $\widehat{a}$ , but  $\widehat{0}$  is all we need.

### 2.2.1 The example of $SU(9)/\mathbb{Z}_9$ at level $k = 18$

Since the argument in the general case is slightly involved it may be instructive to see how one can use this equation to determine the charge group in an explicit example. The example we want to study is  $SU(9)/\mathbb{Z}_9$  at level  $k = 18$ . In this case  $M = 9$ , and we want to solve

$$\dim(\lambda) q(\widehat{a}, i) = \sum_{\widehat{b}} \mathcal{N}_{\lambda\widehat{a}} \widehat{b} q(\widehat{b}, i + t(\lambda)) \pmod{9}, \tag{2.8}$$

where  $\mathcal{N}_{\lambda\widehat{a}} \widehat{b}$  is the  $\mathcal{A}^*$  NIM-rep. We now claim:

**Claim 1:**  $q(\widehat{a}, i) = \dim(a)Q_i$  is always a solution  $\pmod{9}$ , for any choice of  $0 \leq Q_i < 9$ ,  $i = 0, 1, \dots, 8$ , provided only that  $Q_i = Q_{i+3} = Q_{i+6} \pmod{3}$ , for  $i = 0, 1, 2$ . (Note that this constraint follows from (2.7) with  $\lambda = \Lambda_3$  and  $\lambda = \Lambda_6$ .)

Let us assume Claim 1 for a moment. Then it is not difficult to see that the charge group is

$$3 \cdot \mathbb{Z}_9 \oplus 6 \cdot \mathbb{Z}_3 \tag{2.9}$$

(*i.e.* 3 copies of  $\mathbb{Z}_9$  and 6 of  $\mathbb{Z}_3$ ). Indeed, Claim 1 implies that  $Q_0, Q_1, Q_2$  are unconstrained, apart from being between 0 and 9 (so they give us the  $3 \cdot \mathbb{Z}_9$ ); then the value of say  $Q_3$  is determined  $\pmod{3}$  by  $Q_0$ , so our only freedom is to add 3 or 6 to that value (so each of those  $Q_i$ , for  $3 \leq i < 9$ , contribute a copy of  $\mathbb{Z}_3$ ).

So we want to prove Claim 1, *i.e.* that

$$\dim(\lambda) \dim_C(a) Q_i = Q_{i+t(\lambda)} \sum_{\widehat{b}} \mathcal{N}_{\lambda\widehat{a}} \widehat{b} \dim_C(b) \pmod{9}. \tag{2.10}$$

Of course, the  $\mathcal{A}^*$  charge equation simplifies the right side, and we obtain

$$\dim(\lambda) \dim_C(a) Q_i = \dim(\lambda) \dim_C(a) Q_{i+t(\lambda)} \pmod{9}. \tag{2.11}$$

But  $\dim_C(a)$  are integers, and  $\dim_C(a) = 1$  for  $a = 0$ . So (2.11) is equivalent to

$$\dim(\lambda) Q_i = \dim(\lambda) Q_{i+t(\lambda)} \pmod{9}. \tag{2.12}$$

As we know, it is sufficient to consider only the generators  $\lambda$  of the fusion ring, *i.e.* the fundamental weights  $\Lambda_j$  for  $1 \leq j \leq 8$ . We also know  $\dim(\Lambda_j)$  is the binary coefficient  $\binom{9}{j}$ . One can then easily check these fundamental weights one by one to see that (2.12) is indeed satisfied by the  $Q_i$  in Claim 1.

### 2.2.2 The general argument

Now we turn to the general argument. Recall from section 4.1 of [26] the following two facts:

**Proposition 1.** [26] Consider any  $\mathfrak{su}(n)$  ( $n$  can be even), and any prime power  $p^\nu > 1$  exactly dividing  $n$ . Then

- (a)  $\gcd_\lambda \dim(\lambda) = \gcd_\ell \dim(\Lambda_\ell) = p^\nu$ , where we run over all weights  $\lambda \in P_+(\mathfrak{su}(n))$  with  $t(\lambda)$  coprime to  $p$ , and all  $0 < \ell < n$  coprime to  $p$ .
- (b)  $\dim(\Lambda_{p\ell}) = \dim(\bar{\Lambda}_\ell) \pmod{p^\nu}$ , where  $0 < \ell < n/p$  is arbitrary, and  $\Lambda_{p\ell}$  resp.  $\bar{\Lambda}_\ell$  is a fundamental weight of  $\mathfrak{su}(n)$  resp.  $\mathfrak{su}(n/p)$ .

When we say  $d$  exactly divides  $n$ , we mean  $n/d$  is an integer coprime to  $d$ . We write  $d|n$  for a divisor and  $d||n$  for an exact divisor. The dimension of a fundamental weight  $\Lambda_j$  of  $\mathfrak{su}(n)$  is  $\binom{n}{j}$ . We need to generalise part (a) of this proposition. In the following, by e.g.  $\gcd(d^\infty, n)$  we mean  $\prod_{p|d} p^\nu$  where  $p^\nu || n$ .

**Proposition 2.** Again let  $n \geq 2$  be arbitrary. Then

- (a) For any divisors  $d, e$  of  $n$ , with  $e$  dividing  $d^\infty$ , define  $\bar{D} \equiv \gcd(d^\infty, n)$  and  $\bar{D}_e \equiv \gcd_\lambda \dim(\lambda)$ , where we run over all weights  $\lambda \in P_+(\mathfrak{su}(n))$  with  $\gcd(t(\lambda), \bar{D}) = e$ . Then  $\bar{D}_e = \bar{D}/e$ .
- (b) Let  $d$  be any divisor of  $n$ . Then  $\gcd_m \dim(\Lambda_m)$ , as  $m$  ranges over all numbers coprime to  $d$ , is  $\gcd(d^\infty, n)$ .

The proof of Proposition 2(a) is slightly involved; it is given in appendix A. The proof of Proposition 2(b) is a straightforward simplification of that argument.

For any fixed  $0 \leq i < j < d$ , we defined  $e = \gcd(d, j - i)$ . Then (at least when  $n$  is odd) eq. (2.7) tells us

$$\dim(\lambda) \left( q(\widehat{0}, i) - q(\widehat{0}, j) \right) = 0 \pmod{M} \tag{2.13}$$

for all  $\lambda$  with  $\gcd(t(\lambda), d) = e$ . Unfortunately, in Proposition 2(a) we have a slightly different gcd condition on the  $t(\lambda)$ , and it can make a difference. For each prime  $p$  dividing  $e$ , let  $p^\epsilon || e$  as before, and put  $e' = e$  unless  $p^\epsilon || d$ , in which case put  $e' = \nu$ . (Recall that  $p^\nu || n$ .) Now replace  $e$  with its multiple  $e' \equiv \prod_{p|e} p^{\epsilon'}$ . When  $d = \bar{D}$  then  $e' = e$ ; otherwise  $e'$  may be larger than  $e$ .

Now we can use Proposition 2(a). For fixed  $0 \leq i < j < d$  we set  $e = \gcd(d, j - i)$  and define  $e'$  as explained in the previous paragraph. If  $\lambda$  has the property that  $\gcd(t(\lambda), d) = e$

then  $\gcd(t(\lambda), \bar{D})$  divides  $e'$ , and will equal  $e'$  for some  $\lambda$ . Thus Proposition 2(a) implies that for each such  $i < j$

$$q(\widehat{0}, i) = q(\widehat{0}, j) \pmod{M/D_{e'}}, \quad (2.14)$$

where  $D_{e'} \equiv \gcd(\bar{D}/e', M)$ .

**Result:** The general charge solution (2.5), for  $n$  odd, for  $\mathcal{D}^*$  is

$$q(\widehat{a}, i) = \dim_C(a) \left( Q + \frac{M}{D} Q_i \right), \quad (2.15)$$

valid with  $M$  in (1.13). Here  $D = \gcd(\bar{D}, M)$ , and  $0 \leq Q < M$  and  $0 \leq Q_i < D$  are arbitrary except that  $Q_0 = 0$  and  $Q_i = Q_j \pmod{D/D_{e'}}$ , where  $e'$  depends on  $i < j$  as explained above. Hence the charge group for  $\mathcal{D}^*$  is (for odd  $n$ )

$$\mathbb{Z}_M \oplus \bigoplus_{p|\gcd(d,M)} \bigoplus_{i=1}^{\delta} (p^i - p^{i-1}) \cdot \mathbb{Z}_{p^{\min\{\mu, \nu-i+1\}}} \quad (2.16)$$

where  $p^\nu \| n$ ,  $p^\delta \| d$  and  $p^\mu \| M$ . The first sum runs over all primes  $p$  dividing both  $d$  and  $M$ .

For example, when  $d$  is coprime to  $M$ , we get a charge group of  $\mathbb{Z}_M$ . When  $d$  is a prime  $p$  dividing  $M$ , then the charge group is  $\mathbb{Z}_M \oplus (p-1) \cdot \mathbb{Z}_D$ . For  $SU(9)/\mathbb{Z}_9$  at  $k=18$ , we get the charge group  $3 \cdot \mathbb{Z}_9 \oplus 6 \cdot \mathbb{Z}_3$ , as in the analysis of the previous section.

The structure of this charge group for  $\mathcal{D}^*$  ( $n$  odd) is that there is a ‘constant’ solution (*i.e.* one independent of  $i$ ), defined mod  $M$  (this accounts for the left-most summand  $\mathbb{Z}_M$ ); in addition, there are solutions depending on  $i$ , and they are uniquely determined by the ‘vacuum’ values  $q(\widehat{0}, i)$  subject only to the constraint (2.7).  $Q_i$  in (2.15) gives the adjustment of  $q(\widehat{a}, i)$  from  $q(\widehat{a}, 0)$ , which is why we take  $Q_0 = 0$ . Eq. (2.16) builds up the charge group prime by prime; the  $i$ th summand there concerns the value of  $Q_j$  for  $p^{\delta-i} \leq j < p^{\delta-i+1}$ .

The proof of ‘Result’ is now easy (though as mentioned earlier it assumes Conjecture B of [15]). Eq. (2.14) gives an upper bound for the charge group, because  $q(\widehat{0}, i)$  uniquely determines the charge assignment. To see that the proposed charges satisfy (2.5), use the fact that  $\dim_C(a)$  solve the charge equations for  $\mathcal{A}^*$  to reduce (2.5) to

$$\dim(\lambda) \dim_C(a) \frac{M}{D} Q_i = \dim(\lambda) \dim_C(a) \frac{M}{D} Q_{i+t(\lambda)} \pmod{M}. \quad (2.17)$$

Let  $e = \gcd(d, t(\lambda))$ , then  $D_{e'}$  divides  $\dim(\lambda)$  by Proposition 2(a), and eq. (2.17) is satisfied because of the constraint on the  $Q_i$ .

The Chinese Remainder Theorem permits us to treat distinct primes independently, *i.e.* we are only interested in the independent values of each  $Q_i$ , modulo each sufficiently large prime power. So fix a prime  $p|d$ , and write  $p^\alpha \| D$ . Then relation (2.14) says that, modulo  $p^\alpha$ , the  $Q_i$ , for  $0 \leq i < p^\delta$ , determine all remaining  $Q_j$ ,  $p^\delta \leq j < d$ , since  $D_{e'}$  is coprime to  $p$  when  $p^\delta$  divides  $e$ . So those  $Q_j$  can be ignored, as far as  $p$  is concerned. Also, the  $Q_i$ , for  $0 \leq i < p^{\delta-1}$ , determine the  $Q_j$ , for  $p^{\delta-1} \leq j < p^\delta$ , up to an ambiguity

of  $\mathbb{Z}_{D_p} = \mathbb{Z}_{p^{\min\{\mu, \nu+1-\delta\}}}$ . This contributes  $(p^\delta - p^{\delta-1}) \cdot \mathbb{Z}_{p^{\min\{\mu, \nu+1-\delta\}}}$  to the charge group — one  $\mathbb{Z}_{D_p}$  for each  $j$ . Continuing in this way, we get the desired expression for the charge group.

Since  $\mathbb{Z}_a \oplus \mathbb{Z}_b \cong \mathbb{Z}_{ab}$  whenever  $a$  and  $b$  are coprime, the  $\mathcal{D}^*$  charge group can be rewritten in the form (1.14), where  $M_1 = M$ ,  $t = d$ , and  $M_2, \dots, M_d$  all divide  $\gcd(M, d^\infty, n)$ . In fact, most of these  $M_i$  will usually equal 1: the number of non-trivial  $\mathbb{Z}_{M_i}$  will be  $\max_{p|\gcd(d, M)} p^\delta$ .

### 3. The $\mathcal{D}^*$ case with $n$ even, non-pathological

In the previous section we computed the NIM-rep, charges and charge group for the  $\mathcal{D}^*$  NIM-rep when  $n$  is odd. Even  $n$  is more subtle. In particular, we learnt in [26] that (as far as  $\mathcal{D}$  is concerned) there are two cases to distinguish:  $d' \equiv n/d$  even, which behaves almost as simply as  $n$  odd did, and is called *non-pathological*; and  $d'$  odd, which behaves much more peculiarly and is called *pathological*. In the case of  $\mathcal{D}$ , the pathological behaviour was ultimately due to  $\dim(J^{d'}) = -1 \pmod{M}$ ; for  $\mathcal{D}^*$  this distinction remains significant, though there is a different combinatorial cause.

In this section, we try to mimic the analysis of the previous section, but for non-pathological even  $n$ . So  $d$  can be any divisor of  $n$ , provided  $d'$  is even. We will find that we can be nearly as successful here as we were for odd  $n$ .

#### 3.1 The NIM-rep

Write  $n = 2m$ . The NIM-rep for  $d = 1$  (i.e.  $\mathcal{N}(\mathcal{A}^*)$ ) was given in section 4.1.2 of [20]. The exponents of  $\mathcal{A}^*$  consist of all  $\mu \in P_+^k(\mathfrak{su}(n))$  with  $C\mu = \mu$ , all with multiplicity 1. The boundary states  $\widehat{a}$  are parametrised by the level  $k$  integrable highest weights of the twisted affine algebra  $A_{2m-1}^{(2)}$ , i.e. they are all  $(m+1)$ -tuples  $(a_0; a_1, \dots, a_m)$  of non-negative integers where  $k = a_0 + a_1 + 2a_2 + \dots + 2a_m$ . The  $\psi$ -matrix, diagonalising the NIM-rep, is the modular  $\widehat{S}$ -matrix of  $A_{2m-1}^{(2)}$  (see eq. (4.5) in [20]). The NIM-rep coefficients are  $C$ -twisted fusion coefficients of  $A_{n-1}^{(1)}$  level  $k$ , and can be expressed in terms of ordinary fusions of  $B_m^{(1)}$  level  $k+1$  [15]. Thanks to the order-2 symmetry of the Dynkin diagram of the ‘orbit Lie algebra’  $D_{m+1}^{(2)}$ , the NIM-rep has a grading:

$$\mathcal{N}(\mathcal{A}^*)_{\lambda \widehat{a}} \neq 0 \implies t(\lambda) + Q(\widehat{a}) = Q(\widehat{b}) \pmod{2}, \tag{3.1}$$

where  $t(\lambda) = \sum_{i=1}^{n-1} i\lambda_i$  as usual and  $Q(\widehat{a}) = \sum_{i=1}^m ia_i$ . Because of the order-2 symmetry of the Dynkin diagram of the twisted algebra  $A_{2m-1}^{(2)}$ , the NIM-rep has a symmetry:

$$\mathcal{N}(\mathcal{A}^*)_{\lambda \widehat{a}} = \mathcal{N}(\mathcal{A}^*)_{\lambda, K\widehat{a}}^{K\widehat{b}}, \tag{3.2}$$

where  $K(a_0; a_1, a_2, \dots, a_m) = (a_1; a_0, a_2, \dots, a_m)$ ; see section 5.3 of [20] for more details. These are special features of  $n$  being even, and will play a crucial role in the following.

We found in [15] that the charge group for  $\mathcal{A}^*$  is  $\mathbb{Z}_M$ , just as it is for  $\mathcal{A}$ , and the charge  $q(\widehat{a})$  can be taken to be  $\dim_C(a)$ . These facts, which we need below, could only be proven once we assumed Conjectures B and  $B^{\text{spin}}$  (see section 1.2 of [15] for details). These conjectures seem quite safe.

If  $n$  and  $d'$  are even the exponents of  $\mathcal{D}^*$  form the multi-set  $\bigcup_{j=1}^d \bigcup_{\mu=C\mu} J^{jd'/2}\mu$ . That is,  $\mu \in P_+^k(\mathfrak{su}(n))$  will have multiplicity 0 unless  $C(J^{jd'/2}\mu) = J^{jd'/2}\mu$  for some  $j$ , in which case its multiplicity will equal its order as a  $J^{d'}$ -fixed point. The  $1/2$  in the exponent was necessary in order to move each exponent into a  $C$ -invariant weight. This is a key difference between pathological and non-pathological cases: in the pathological case (*i.e.*  $d'$  odd), not all exponents are related to  $C$ -invariant weights, and so there is no chance of a direct relation between  $\mathcal{N}(\mathcal{D}^*)$  and  $\mathcal{N}(\mathcal{A}^*)$ . The reason we take  $1 \leq j \leq d$  in the union rather than  $1 \leq j \leq 2d$  is because otherwise we would double-count: if  $C\mu = \mu$ , then also  $C(J^{n/2}\mu) = J^{n/2}\mu$ .

The previous construction  $\mathcal{N}(\mathcal{A}^*) \otimes \mathbb{Z}_d$  fails here (at least for  $d$  even), because  $J^{n/2}0$  is an exponent for  $\mathcal{N}(\mathcal{A}^*)$ , so 0 is an exponent of  $\mathcal{N}(\mathcal{A}^*) \otimes \mathbb{Z}_d$  with multiplicity 2 (when  $d$  is even). This means that  $\mathcal{N}(\mathcal{A}^*) \otimes \mathbb{Z}_d$  is decomposable into a direct sum of 2 (isomorphic) NIM-reps (when  $d$  is even). More precisely, thanks to the grading (3.1), the action

$$\lambda.(\widehat{a}, i) = (\lambda.\widehat{a}, t(\lambda) + i) \tag{3.3}$$

obeys the property

$$Q(\lambda.\widehat{a}) + (t(\lambda) + i) = Q(\widehat{a}) + i \pmod{2} . \tag{3.4}$$

Thus the two (indecomposable) summands in  $\mathcal{N}(\mathcal{A}^*) \otimes \mathbb{Z}_d$  (for  $d$  even) consist of the pairs  $(\widehat{a}, i)$  with  $Q(\widehat{a})$  and  $i$  congruent, or not congruent, mod 2.

Instead, the correct  $\mathcal{D}^*$  NIM-rep is an irreducible summand of  $\mathcal{N}(\mathcal{A}^*) \otimes \mathbb{Z}_{2d}$ , for definiteness, say, the one obeying

$$Q(\widehat{a}) = i \pmod{2} . \tag{3.5}$$

This is valid for  $d$  even or odd, provided only that  $d'$  is even. The proof is simple: the exponents of  $\mathcal{N}(\mathcal{A}^*) \otimes \mathbb{Z}_{2d}$  are  $J^{jd'/2}\mu$  for all  $C\mu = \mu$  and  $0 \leq j < 2d$ , and the exponents of the irreducible summand are half of those (*i.e.* divide each multiplicity by 2). Note again the importance of  $d'$  being even:  $\mathcal{N}(\mathcal{A}^*) \otimes \mathbb{Z}_{2d}$  will define a NIM-rep only if  $2d$  divides  $n$ .

When  $d$  is odd, there is a more direct construction of this NIM-rep: it is simply  $\mathcal{A}^* \otimes \mathbb{Z}_d$ . The isomorphism sends  $(\widehat{a}, i) \in \mathcal{A}^* \otimes \mathbb{Z}_d$  to  $(\widehat{a}, dQ(\widehat{a}) + (1-d)i) \in \mathcal{A}^* \otimes \mathbb{Z}_{2d}$ . Then

$$\begin{aligned} \lambda.(\widehat{a}, dQ(\widehat{a}) + (1-d)i) &= (\lambda.\widehat{a}, dQ(\widehat{a}) + (1-d)i + t(\lambda)) \\ &= (\lambda.\widehat{a}, dQ(\lambda.\widehat{a}) + (1-d)(i + t(\lambda))) , \end{aligned} \tag{3.6}$$

which indeed corresponds to  $(\lambda.\widehat{a}, i + t(\lambda))$ , and so the two  $\lambda$ -actions agree. On the other hand, when  $d$  is even,  $1-d$  is invertible mod  $2d$  so this merely defines an automorphism of  $\mathcal{N}(\mathcal{A}^*) \otimes \mathbb{Z}_{2d}$ , rather than a projection onto  $\mathcal{N}(\mathcal{A}^*) \otimes \mathbb{Z}_d$ .

The boundary states of  $\mathcal{D}^*$  can thus be identified with pairs  $(\widehat{a}, i)$  where  $Q(\widehat{a}) = i \pmod{2}$ . The  $\psi$  matrix for the NIM-rep  $\mathcal{N}(\mathcal{A}^*) \otimes \mathbb{Z}_{2d}$  is given as in the previous section, with  $d$  replaced everywhere with  $2d$ . The  $\psi$ -matrix for  $\mathcal{D}^*$  is then

$$\psi(\mathcal{D}^*)_{(\widehat{a}, i), (\mu, j)} = \frac{1}{\sqrt{d}} \exp[\pi i ij/d] \widehat{S}_{\widehat{a}, \mu} , \tag{3.7}$$

where  $\mu = C\mu$ ,  $0 \leq i < 2d$ ,  $i = Q(\widehat{a}) \pmod{2}$ , and  $1 \leq j \leq d$ . The restriction of the rows (*i.e.* of  $i$ ) is clear; the restriction of the columns is because the  $(K\widehat{a}, j)$  column now equals the  $(\widehat{a}, j + d)$  column. As before, this amounts to specifying the exponents of  $\mathcal{D}^*$  by pairs  $(\mu, j)$ , where  $\mu = C\mu$  and  $1 \leq j \leq d$ .

### 3.2 Charges and charge groups

Having given a fairly explicit description of  $\mathcal{N}(\mathcal{D}^*)$  for the non-pathological ( $d'$  even) case for  $n$  even, we can now attempt to determine the corresponding charge group. As we have seen above, the structure of the NIM-rep depends on whether  $d$  is even or not.

If  $d$  is odd, then  $\mathcal{N}(\mathcal{D}^*)$  has in fact the same structure as for  $n$  odd, and the charges and charge group is given as in the ‘Result’ of the previous section. Indeed, the identical arguments apply.

If  $d$  is even, on the other hand, the structure of the NIM-rep is more complicated, and so is the analysis of its charges. As in section 2.2, the charges are uniquely determined by the values  $q(\widehat{0}, i)$ . Also, taking  $q(\widehat{a}, i) = \dim_C(a)$  defines an order  $M$  solution. Our proof of the Result in section 2 followed because constraint (2.7) there gives an upper bound for the charge group, while the charge ansatz (2.15) gives a lower bound, and the two agree. Constraint (2.7) now implies eq. (2.14) as before; although the 2 in (2.7) is no longer invertible mod  $d$ , it is now superfluous because of the parity condition on the  $i$ . However the ansatz (2.15) now in general works only mod  $M/2$ .

That ansatz does not respect (3.5); a more natural one is to introduce parameters  $Q_{i,s}$  for  $0 \leq i < 2d$  and  $s = 0, 1$  where

$$q(\widehat{a}, i) = \dim_C(a) \frac{M}{D} Q_{i, Q(\widehat{a})} . \tag{3.8}$$

Now we can demand  $Q_{i,s} = 0$  unless  $i = s \pmod{2}$ . Unfortunately, this ansatz fails no better: it solves the charge equation provided we take  $Q_{i,0} = Q_{j,0} \pmod{D/D_{e'}}$ , where we restrict to even  $i, j$ , and define  $e'$  using  $2d$  rather than  $d$ . The result is that the  $\mathcal{D}^*$  charge group contains

$$\mathbb{Z}_M \oplus \bigoplus_{i=1}^{\delta} (2^i - 2^{i-1}) \cdot \mathbb{Z}_{2^{\min\{\mu, \nu-i\}}} \oplus \bigoplus_{\substack{p|\gcd(d,M) \\ p \neq 2}} \bigoplus_{i=1}^{\delta} (p^i - p^{i-1}) \mathbb{Z}_{p^{\min\{\mu, \nu-i+1\}}} . \tag{3.9}$$

We also know from (2.14) that it is contained in the charge group (2.16).

The  $\mathcal{D}^*$ -charge group is a *proper* subgroup of (2.16) iff (2.7) can be supplemented by new constraints on  $q(\widehat{0}, i)$ ; it *properly* contains (3.9) iff a more general ansatz than (3.8) can be found and is effective. When there are at least as many 2’s dividing  $n$  as  $dM$ , the two bounds agree.

We conjecture that the  $\mathcal{D}^*$ -charge group is *always* given by (2.16). While we do not have a general argument for this assertion, we have checked it explicitly for the simplest non-trivial case  $SU(4)/\mathbb{Z}_2$  (see below). Also, if the  $\mathcal{D}^*$ -charge group is to agree with the  $\mathcal{D}$ -charge group (as it appears to do), then the analysis of the next section implies that the  $\mathcal{D}^*$ -charge group must be (2.16). We regard this as convincing evidence that the  $\mathcal{D}^*$ -charge group is indeed given by (2.16).

For the example of  $SU(4)/\mathbb{Z}_2$  the two bounds (2.16), (3.9) agree, and therefore equal the  $\mathcal{D}^*$ -charge group, unless  $k$  is an odd multiple of 4, so it suffices to restrict attention here to the latter. The boundary labels are  $([a_1, a_2], j)$  where  $0 \leq j < 4$ ,  $a_i \geq 0$ , and  $a_1 + 2a_2 \leq k$ . The NIM-rep is built out of the  $\mathcal{A}^*$  one in the way described above; the  $\mathcal{A}^*$  NIM-rep is generated by

$$\begin{aligned} \Lambda_1.[a_1, a_2] &= \Lambda_3.[a_1, a_2] = [a_1 + 1, a_2] + [a_1 - 1, a_2] + [a_1 - 1, a_2 + 1] + [a_1 + 1, a_2 - 1] \\ \Lambda_2.[a_1, a_2] &= 2[a_1, a_2] + [a_1, a_2 - 1] + [a_1 - 2, a_2 + 1] + [a_1 + 2, a_2 - 1] + [a_1, a_2 + 1], \end{aligned}$$

where we drop any term  $[a'_1, a'_2]$  with a component  $a'_1, a'_2$ , or  $a'_0 \equiv k - a'_1 - 2a'_2$  equal to  $-1$ . In addition, when  $a_1 = 0$  use  $[-2, a_2 + 1] = -[0, a_2]$  and when  $k = a_1 + 2a_2$  use  $[a_1, a_2 + 1] = -[a_1, a_2]$ .

We have checked by an explicit computation that for all such  $k$  the charge group is  $\mathbb{Z}_M \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$ , in agreement with (2.16). The solution of order  $M$  is simply  $q(\widehat{a}, j) = \dim_C(\widehat{a})$ ; the order 4 assignments are distinguished by their values of  $q(\widehat{0}, 0)$  and  $q(\widehat{0}, 2)$ , and have the special property that the charges  $q([a_1, a_2], j)$  are periodic in both  $a_1, a_2$  with period 8. (Given this periodicity, one can check the existence of this solution fairly straightforwardly for all such  $k$ .)

#### 4. The $\mathcal{D}$ charge group in the non-pathological case

We now want to compare the results of the previous two sections with the charge analysis for the  $\mathcal{D}$  theory. Partial results for that were already found in [26], but now we are able to go several steps further, at least in the non-pathological case.

##### 4.1 The NIM-rep

The modular invariant here is given by (1.11), and requires that  $n(n+1)k/d$  be even. Write  $f = \gcd(d, k)$ . Explicitly listing the exponents requires distinguishing two cases:

**Case A:** Either  $n$  or  $k$  is odd, or either  $n/d$  or  $k/f$  is even. Then  $\mu \in P_+^k(\mathfrak{su}(n))$  is an exponent if and only if  $d$  divides  $t(\mu)$ . Such a  $\mu$  has multiplicity  $o(\mu)$ .

**Case B:** Both  $n$  and  $k$  are even, and both  $n/d$  and  $k/f$  are odd. Then the exponents come in two versions (see [26] for details). Case B is necessarily pathological, and we shall not consider it in this paper.

The boundary states  $a \in \mathcal{B}$  here correspond to pairs  $([\nu], i)$ , where  $[\nu] = \{J^{jd} \nu\}$  is the  $J^{d'}$ -orbit of any weight  $\nu \in P_+^k(\mathfrak{su}(n))$ , and where  $1 \leq i \leq o(\nu)$ . To simplify notation, we will write  $[\nu, i]$  for  $([\nu], i)$ , and when  $\nu$  has order  $o(\nu) = 1$ , then we shall usually write  $[\nu]$  instead of  $([\nu], 1) = [\nu, 1]$ .

We will restrict attention in this paper to Case A: it contains all non-pathological cases, as well as some pathological ones. The  $\psi$ -matrix is given by (B.1), and the NIM-rep  $\mathcal{N}(\mathcal{D})$  is thus obtained from (1.12). More explicitly, it can be described as follows. When  $\nu$  is not a fixed point, (1.17) of [26] reads

$$\mathcal{N}(\mathcal{D})_{\lambda[\nu]}^{[\nu', i]} = \sum_{j=1}^{d/o(\nu')} N_{\lambda\nu} J^{d'j\nu'} \tag{4.1}$$

for any weight  $\lambda \in P_+^k(\mathfrak{su}(n))$  and boundary label  $[\nu', i]$ . More generally, for any weight  $\lambda \in P_+^k(\mathfrak{su}(n))$  and boundary labels  $[\nu, i], [\nu', i']$ , (1.18) of [26] reads

$$\sum_{i=1}^{o(\nu)} \mathcal{N}(\mathcal{D})_{\lambda[\nu, i]}^{[\nu', i']} = \sum_{j=1}^{d/o(\nu')} N_{\lambda\nu} J^{d'j\nu'} . \tag{4.2}$$

Computing the remaining NIM-rep entries is much more subtle, but in appendix B we obtain these for the fundamental weights and simple currents. (Recall that the NIM-rep matrices for the fundamental weights determine all NIM-rep matrices uniquely.) The result is an unexpectedly simple generalisation of (4.1):

$$\mathcal{N}(\mathcal{D})_{\Lambda_m, [\varphi, i]}^{[\varphi', i']} = \sum_{j=1}^{d/g''} \delta_{ii'}^{(\Delta)} N_{J^{d'j\Lambda_m/\Delta, \varphi^\Delta} }^{\Delta} \varphi'^{\Delta} \tag{4.3}$$

$$\mathcal{N}(\mathcal{D})_{J0, [\varphi, i]}^{[\varphi', i']} = \delta_{[\varphi'], [J\varphi]} \delta_{i, i'} , \tag{4.4}$$

where again we interpret the right-side of (4.3) as vanishing if  $\Delta$  does not divide  $m$ . Here,  $g'' = \text{lcm}(o(\varphi), o(\varphi'))$  and  $\Delta = \text{gcd}(o(\varphi), o(\varphi'))$ .  $N^{(\Delta)}$  stands for the fusion coefficients for  $\widehat{\mathfrak{su}}(n/\Delta)_{k/\Delta}$ .  $\varphi^\Delta$  means to truncate the  $n$ -tuple  $\varphi$  after  $n/\Delta$  components, and to regard it as a weight of  $\mathfrak{su}(n/\Delta)$ . We verify these formulae in appendix B, using the fixed point factorisation formulae of [30]. This new formula (4.3) is why we can now say much more about the  $\mathcal{D}$ -charges than we could in [26].

## 4.2 Charges and charge groups

With our improved understanding of the NIM-rep for  $\mathcal{D}$ , we can now do much better at determining its charge group than in the original analysis of [26]. For concreteness we will first discuss the case of  $\text{SU}(9)/\mathbb{Z}_9$  at level  $k = 18$ .

### 4.2.1 The example of $\text{SU}(9)/\mathbb{Z}_9$ at level $k = 18$

The new key ingredient are the ‘fixed point factorisation’ formulae, expressing the  $\mathcal{D}$ -NIM-rep at fixed points in terms of  $\widehat{\mathfrak{su}}(3)_6$  fusions. The interesting NIM-rep entries are (4.3):

- (i) when  $3|m$  and the order of fixed points  $\phi, \psi$  are both 3, then

$$\mathcal{N}(\mathcal{D})_{\Lambda_m, [\phi, i]}^{[\psi, j]} = \sum_{h=1}^3 \delta_{ij} \bar{N}_{\bar{J}^h \bar{\Lambda}_{m/3}, \bar{\phi}}^{\bar{\psi}} . \tag{4.5}$$

- (ii) when  $3|m$  and the order of fixed points  $\phi, \psi$  are 3 and 9, then

$$\mathcal{N}(\mathcal{D})_{\Lambda_m, [\phi, i]}^{[\psi, j]} = \delta_{ij}^{(3)} \bar{N}_{\bar{\Lambda}_{m/3}, \bar{\phi}}^{\bar{\psi}} . \tag{4.6}$$

We let bars denote  $\widehat{\mathfrak{su}}(3)_6$  quantities. We know already from [26] that the ‘unresolved’ charge group for  $\mathcal{D}$  is  $\mathbb{Z}_9$ . So we only need to determine the ‘resolved’ charge group.<sup>2</sup>

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<sup>2</sup>In [26] we called the ‘unresolved’ and ‘resolved’ charge groups ‘untwisted’ and ‘twisted’, respectively. In the current context this is bound to lead to confusions! We shall therefore use the terms ‘unresolved’ and ‘resolved’ in this paper.



That is, we can impose the condition that the charges  $q[\lambda]$  of any non-fixed point  $\lambda$  be 0. The remaining charges can be parametrised by  $\mathfrak{su}(3)$  weights — more precisely by pairs  $[\bar{\phi}, i] \equiv ([\bar{\phi}], i)$ , where  $[\bar{\phi}]$  is a  $\bar{J}$ -orbit in  $\widehat{\mathfrak{su}}(3)_6$ , and  $1 \leq i \leq 3$  if  $\bar{\phi}$  is not the fixed point (22), while  $1 \leq i \leq 9$  if  $\bar{\phi} = (22)$ . We need to understand the complete list of constraints on these ‘resolved’ charges  $q[\bar{\phi}, i]$ .

One constraint is easy. Let  $\phi$  be the  $\widehat{\mathfrak{su}}(9)_{18}$  fixed point  $(\bar{\phi}, \bar{\phi}, \bar{\phi})$  corresponding to the  $\widehat{\mathfrak{su}}(3)_6$  weight  $\bar{\phi}$ , and write  $\bar{0}$  for  $(0, 0)$ . Then we know from [26] (or directly from the  $\mathcal{D}(\widehat{\mathfrak{su}}(9)_{18})$  charge equation with  $\lambda = \phi$  and  $a = [\bar{0}, j]$ ) that

$$\sum_j q[\bar{\phi}, j] = 0 \pmod{9}, \tag{4.7}$$

where we sum over  $1 \leq j \leq 3$  for  $\bar{\phi} \neq (2, 2)$ , and over  $1 \leq i \leq 9$  for  $\bar{\phi} = (2, 2)$ .

We know on general grounds that it is sufficient to consider the charge equations only for the fundamental weights, *i.e.* the equations

$$\dim(\Lambda_m) q[\lambda, i] = \sum_{[\mu, j]} \mathcal{N}_{\Lambda_m, [\lambda, i]^{[\mu, j]}} q[\mu, j] \pmod{9}. \tag{4.8}$$

Because we are in the realm of the ‘resolved’ charges, we only have to worry about the charges of fixed points on the right-hand-side, and the relevant NIM-rep coefficients are given by the fixed point factorisation formulae (4.5) and (4.6).

If  $\lambda$  is not an  $\widehat{\mathfrak{su}}(9)_{18}$  fixed point, then by assumption  $q[\lambda, i] = 0$ , so the right-hand-side sum must also be 0 (mod 9). But the relevant NIM-rep coefficients in this case are given by (4.1), and so in particular the right-hand-side of that charge equation will involve sums as in (4.7). Thus the charge equation (4.8) when  $\lambda$  is not an  $\widehat{\mathfrak{su}}(9)_{18}$  fixed point are automatically satisfied as long as (4.7) is satisfied; they therefore do not supply any new constraints.

So all we have to consider are  $\dim(\Lambda_m)q[\bar{\phi}, i]$ . If 3 does not divide  $m$ , then 9 will divide  $\dim(\Lambda_m)$  (by Proposition 1 — see section 2.2.2), so the left-hand-side of (4.8) will be 0. But so will the right-hand-side since 3 will not divide the ‘triality’ of  $\Lambda_m$  (namely  $m$ ) plus the triality of  $\phi$  (namely  $3t(\bar{\phi}) + 9k$ ), so no fixed points can appear on the right-hand-side. Thus the charge equation is trivially satisfied.

So all we really have to consider are  $\dim(\Lambda_m)q[\bar{\phi}, i]$  when  $m = 3$  or  $m = 6$ . In this case the left-hand-side of (4.8) becomes  $3q[\bar{\phi}, i] \pmod{9}$ . Using the fixed point factorisation formulae (4.5) and (4.6), the right-hand-side becomes

$$\sum_{[\bar{\psi}] \neq (22)} \sum_{h=1}^3 \bar{N}_{\bar{J}^h \bar{\Lambda}_{m/3, \bar{\phi}}^{\bar{\psi}}} q[\bar{\psi}, i] + \bar{N}_{\bar{\Lambda}_{m/3, \bar{\phi}}^{(22)}} \sum_{h=1}^3 q[(22), i + 3h] \pmod{9} \tag{4.9}$$

provided that  $\bar{\phi} \neq (22)$ , and

$$\sum_{[\bar{\psi}] \neq (22)} \bar{N}_{\bar{J}^h \bar{\Lambda}_{m/3, \bar{\phi}}^{\bar{\psi}}} q[\bar{\psi}, i] \pmod{9} \tag{4.10}$$

if  $\bar{\phi} = (22)$ . Here, we have assumed for notational convenience that  $q[\bar{\psi}, i]$  is periodic in  $i$  with period 3 (for  $\bar{\psi} \neq (22)$ ) or 9 (for  $\bar{\psi} = (22)$ ).

The point of all this is, that we are to recognise these NIM-rep coefficients (for  $\mathcal{D}(\widehat{\text{su}}(9)_{18})$  involving 2 fixed points) as precisely the NIM-rep coefficients for  $\mathcal{D}(\widehat{\text{su}}(3)_6)$ . That is, these last charge equations are indistinguishable from those for  $\mathcal{D}(\widehat{\text{su}}(3)_6)$ !

Collecting all this together, we obtain the statement that the ‘resolved’ charge equations for the  $\mathcal{D}$  charge group of  $\text{SU}(9)/\mathbb{Z}_9$ ,  $k = 18$ , consist of (4.7), together with 3 decoupled copies of the whole collection of charge equations for the  $\mathcal{D}$  charge group of  $\text{SU}(3)/\mathbb{Z}_3$ ,  $k = 6$ . The 3 decoupled equations apply to the 3 values of  $i \pmod{3}$ ; the  $\mathcal{D}$  charge equations for  $\text{SU}(3)/\mathbb{Z}_3$  were analysed in detail in section 2 of [26].

Ignoring (4.7) temporarily, this means we would get a net resolved charge group of 3 copies of the  $\mathcal{D}$  charge group of  $\widehat{\text{su}}(3)_6$ , i.e.

$$3 \cdot (\mathbb{Z}_9 \oplus 2 \cdot \mathbb{Z}_3) = 3 \cdot \mathbb{Z}_9 \oplus 6 \cdot \mathbb{Z}_3 .$$

The 3  $\mathbb{Z}_9$ ’s correspond to the values of  $q[\bar{0}, i]$  ( $1 \leq i \leq 3$ ); the 6  $\mathbb{Z}_3$ ’s correspond to the values of  $q[(22), j]$  ( $1 \leq j \leq 6$ ). More precisely, the values of  $q[\bar{0}, i]$  can take any value from 0 to 8, whereas  $q[(22), j]$  are constrained by (2.14) and (2.16) of [26], which say that

$$q[(22), j] + q[(22), j + 3] + q[(22), j + 6] = 0 \pmod{9} \quad (4.11)$$

and

$$3q[(22), j] = 6 \pmod{9}, \quad (4.12)$$

respectively. The first equation (4.11) fixes  $q[(22), 7]$ ,  $q[(22), 8]$ ,  $q[(22), 9]$  in terms of  $q[(22), j]$  with  $1 \leq j \leq 6$ . The second (4.12) implies that  $q[(22), j] = 2 \pmod{3}$ , which fixes the remaining  $q[(22), j]$  up to a  $\mathbb{Z}_3$  ambiguity.

This leaves us with imposing (4.7). For  $\bar{\phi} = (22)$  we get simply the constraint (4.11) again. So this means that we keep the 6  $\mathbb{Z}_3$ ’s. But (4.7) for  $\bar{0}$  is non-trivial, and fixes  $q[\bar{0}, 3] = -q[\bar{0}, 1] - q[\bar{0}, 2]$ . This drops the  $3 \cdot \mathbb{Z}_9$  to  $2 \cdot \mathbb{Z}_9$ . Then (4.7) will be automatically satisfied for the remaining  $\bar{\phi}$ , since  $q[\bar{\phi}, i]$  will equal  $\text{dim}(\bar{\phi})q[\bar{0}, i]$ .

Thus the resolved  $\mathcal{D}$  charge group for  $\text{SU}(9)/\mathbb{Z}_9$  level  $k = 18$  is in fact  $2 \cdot \mathbb{Z}_9 \oplus 6 \cdot \mathbb{Z}_3$ . Together with the unresolved charge group  $\mathbb{Z}_9$ , the total  $\mathcal{D}$  charge group is therefore

$$3 \cdot \mathbb{Z}_9 \oplus 6 \cdot \mathbb{Z}_3 . \quad (4.13)$$

This agrees precisely with the result for the  $\mathcal{D}^*$  charge group, eq. (2.9).

More generally, the identical argument applies for  $\text{SU}(9)/\mathbb{Z}_9$  whenever 9 divides the level  $k$ , in which case we obtain the resolved charge group  $2 \cdot \mathbb{Z}_M \oplus 6 \cdot \mathbb{Z}_3$ . Together with the unresolved charge group  $\mathbb{Z}_M$ , this means that the  $\mathcal{D}$  and  $\mathcal{D}^*$  charge groups for  $\text{SU}(9)/\mathbb{Z}_9$  when 9 divides  $k$ , are both

$$3 \cdot \mathbb{Z}_M \oplus 6 \cdot \mathbb{Z}_3 . \quad (4.14)$$

#### 4.2.2 The general argument

For general  $n$  (Case A), the same things happen: in particular, the resolved charge equations for  $\text{SU}(n)/\mathbb{Z}_d$  can be expressed in terms of the full charge equations of  $\text{SU}(n/\Delta)/\mathbb{Z}_{d/\Delta}$  for some  $\Delta$ , using fixed point factorisation. This means that the resolved charge group for

$SU(n)/\mathbb{Z}_d$  can be expressed in terms of (un)resolved charge groups for  $SU(n/\Delta)/\mathbb{Z}_{d/\Delta}$ , although a non-trivial amount of book-keeping has to be done to arrive at the final answer.

The full  $\mathcal{D}$ -charge group appears to be given by (2.16), but we do not have a general proof of this yet. The structure of this charge group says that there is a charge assignment given by  $q([0]) = 1$  defined mod  $M$  which accounts for the left-most summand  $\mathbb{Z}_M$ ; it corresponds to the ‘unresolved’ charges (that were called ‘untwisted’ in [26]). It obeys  $q([\lambda]) = \dim(\lambda)$  on non-fixed points, but is not uniquely defined on the fixed points. That ambiguity is completely captured by what we call the ‘resolved charges’ (that were called ‘twisted charges’ in [26] — charge assignments with  $q([0]) = 0$ ). We will find that resolved charges are more tractable than unresolved ones. The charge assignments are uniquely determined by the values of  $q([0])$  and  $q([0_{p^i}, j])$ , where  $0_{p^i} = (k/p^i; 0, \dots, 0, k/p^i, 0, \dots)$  with a  $k/p^i$  placed in every  $n/p^i$ th entry. Here  $1 \leq i \leq \delta$  with  $p^\delta \parallel d$ , and  $1 \leq j \leq p^i$ . The fixed point  $0_{p^i}$  generates all order  $p^i$  fixed points, and corresponds to the vacuum in  $\widehat{su}(n/p^i)_{k/p^i}$ . All possible charge assignments can be explicitly built up inductively from those of the prime-power orbifolds  $SU(n/p^i)/\mathbb{Z}_{p^{\delta-i}}$ . In particular, the summands in (2.16) associated with  $i$  correspond to the freedom in choosing the values of  $q([0_{p^i}, j])$ .

We can prove most of these statements, as we shall see shortly. The main uncertainty is the existence of an unresolved  $\mathcal{D}$ -charge assignment for each  $SU(n/p^i)/\mathbb{Z}_{p^{\delta-i}}$  level  $k/p^i$ . Our arguments hold for Case A (which includes all non-pathological cases as well as some pathological ones). As (2.16) indicates, we will want to look at each prime  $p$  separately — see appendix C for the detailed arguments. Consider first the resolved charges of the  $SU(n)/\mathbb{Z}_d$   $\mathcal{D}$ -theory. As before we define for each prime  $p$ , the parameters  $\nu$ ,  $\mu$  and  $\delta$  by  $p^\nu \parallel n$ ,  $p^\mu \parallel M$  and  $p^\delta \parallel d$ .

The following congruences uniquely specify integers  $q[\phi, i]$  mod  $M$ . For each prime  $p$  dividing both  $M$  and  $d$ , let  $q_p[\phi, i]$  be a resolved charge assignment (mod  $M$ ) for  $SU(n)/\mathbb{Z}_{p^\delta}$  level  $k$ . For a given  $J^{d'}$ -fixed point  $\phi$  with order  $o(\phi)$ , let  $p^\ell \parallel o(\phi)$ . We require that our integers  $q[\phi, i]$  satisfy  $q[\phi, i] = (o(\phi)/p^\ell)^{-1} q_p[\phi, i] \pmod{p^\mu}$  ( $o(\phi)/p^\ell$  is coprime to  $p$  so is invertible mod  $p^\mu$ ). For each prime  $p$  dividing  $M$  but not  $d$ , we also require  $q[\phi, i] = 0 \pmod{p^\mu}$ . This fixes  $q[\phi, i]$  uniquely, and it is not difficult to see that the integers so obtained define a resolved charge assignment for  $SU(n)/\mathbb{Z}_d$  level  $k$ . Moreover, all resolved charge assignments for  $SU(n)/\mathbb{Z}_d$  can be described in this way.

This tells us that for resolved charges, it suffices to consider orbifolds by prime powers. Note that the orbits  $[\phi]$  in the charges  $q_p[\phi, i]$  defined for  $SU(n)/\mathbb{Z}_{p^\delta}$  are with respect to  $J^{n/p^\delta}$ , while the orbits for the actual charges  $q[\phi, i]$  in  $SU(n)/\mathbb{Z}_d$  are with respect to  $J^{n/d}$ . The former are smaller, by a factor of  $o(\phi)/p^\ell$ , which is precisely the origin of the above factor. Eq. (4.4) guarantees that  $q_p[J^{d'}\phi, i] = q_p[\phi, i]$ , so our formula is well-defined.

It thus remains to construct resolved charge assignments for  $SU(n)/\mathbb{Z}_{p^\delta}$  level  $k$ . Choose any  $p$   $\mathcal{D}$ -charge assignments  $q_p^{(h)}[\lambda_p, i]$  for  $SU(n/p)/\mathbb{Z}_{p^{\delta-1}}$  level  $k/p$ , where  $1 \leq h \leq p$ ,  $\lambda_p \in P_+^{k/p}(\mathfrak{su}(n/p))$ , and  $i$  runs from 1 to the  $J^{n/p^{\delta-1}}$ -order of  $\lambda_p$ . These  $p$  charge assignments may be identical or different, but come with an order. We require as well the condition that  $\sum_{h=1}^p q_p^{(h)}[0_p] = 0$ , where the weight  $0_p$  denotes the  $\widehat{su}(n/p)_{k/p}$  vacuum  $0_p = (k/p; 0, \dots, 0)$ . (This condition will guarantee that we end up with resolved charges for  $SU(n)/\mathbb{Z}_d$ .) Then

we get a resolved charge assignment  $q$  for  $SU(n)/\mathbb{Z}_{p^\delta}$  level  $k$ , by defining  $q[\lambda] \equiv 0$  if  $\lambda$  is not fixed by a non-trivial power of  $J^{n/p^\delta}$ , and defining  $q[\phi, (i-1)p+h] \equiv q_p^{(h)}[\phi^p, i]$  for any fixed point  $\phi$ . (For each fixed point  $\phi$  of  $J^{n/p^\delta}$ ,  $\phi^p = \bar{\phi}$  means to truncate it after  $n/p$  components.)

Using this identification, the resolved charges for  $SU(n)/\mathbb{Z}_{p^\delta}$  level  $k$  can be built up recursively from the charges of  $SU(n/p^i)/\mathbb{Z}_{p^{\delta-i}}$  level  $k/p^i$ . Note that in order for this to work we need that  $M(\widehat{su}(n)_k) = M(\widehat{su}(n/p)_{k/p})$ , as is readily verified from (1.13).

The relation of *unresolved* charges for  $SU(n)/\mathbb{Z}_d$  to those for  $SU(n)/\mathbb{Z}_{p^\delta}$  is as for the resolved ones, except that for each prime  $p$  dividing  $M$  but not  $d$ , we now want  $q[\phi, i] = (o(\phi))^{-1} \dim(\phi) \pmod{p^\mu}$  ( $o(\phi)$  will be coprime to  $p$  and hence also invertible mod  $p^\ell$ ). At present we know of no direct relation between unresolved charges of  $SU(n)/\mathbb{Z}_{p^\delta}$  and those of  $SU(n/p)/\mathbb{Z}_{p^{\delta-1}}$ , although fixed point factorisation is surely involved, and identifying those unresolved charges is the only remaining issue here.

We conjecture that an order  $M$  untwisted charge can always be found. If true, this would imply that the  $\mathcal{D}$  charge group is given by (2.16). Infinite classes where this is known to hold (*e.g.*  $M$  coprime to  $d$ ) are given in [26]. Another, generalising the example  $SU(6)/\mathbb{Z}_3$  given in [26], is  $SU(mp)/\mathbb{Z}_p$  where  $1 < m < p$ . It suffices to consider those levels  $k$  such that  $p^2$  divides  $k+mp$  (otherwise  $M$  would be coprime to  $p$ ). We claim that  $p$  will divide any fixed point dimension  $\dim(\phi)$  here, and hence that  $q[\phi, i] = \dim(\phi)/p$  works. This follows directly from Weyl's dimension formula

$$\dim(\phi) = \frac{\left(\frac{k+mp}{p}\right)^{mp(p-1)/2} \left(\prod_{\ell=1}^{p-1} \ell^{m(p-1)} \prod_{1 \leq i < j \leq m} (\ell^2 \left(\frac{k+mp}{p}\right)^2 - (\phi(j) - \phi(i))^2)\right)}{p^{pm(m-1)/2} \left(\prod_{\ell=1}^{m-1} \ell^{p(m-1)} \prod_{1 \leq i < j \leq p} (\ell^2 p^2 - (j-i)^2)\right)} \times \frac{\left(\prod_{1 \leq i < j \leq m} (\phi(j) - \phi(i))^p\right)}{\left(\prod_{1 \leq i < j \leq p} (j-i)^m\right)}, \tag{4.15}$$

where  $\phi(i) \equiv \sum_{l=1}^i \phi_l$ . Counting the occurrences of  $p$ , we get at least  $mp(p-1)/2$  in the numerator and exactly  $pm(m-1)/2$  in the denominator. Thus for  $SU(mp)/\mathbb{Z}_p$  when  $1 < m < p$ , at any level  $k$ , the  $\mathcal{D}$  charge group is  $\mathbb{Z}_M \oplus (p-1) \cdot \mathbb{Z}_p$  or  $\mathbb{Z}_M$ , depending on whether or not  $p^2$  divides  $k+mp$ , in perfect agreement with (2.16).

### 5. Comparing charge groups of $\mathcal{D}$ and $\mathcal{D}^*$

As we have mentioned before, the charge groups for  $\mathcal{A}$  and  $\mathcal{A}^*$  are known to agree [15–17] in all cases. We have furthermore seen evidence in the above that the same is true for the charge groups for  $\mathcal{D}$  and  $\mathcal{D}^*$  — indeed our two conjectures imply that they will always agree in the non-pathological cases, and be given by (2.16). From the point of view of this paper this agreement is quite remarkable, considering that the analysis for  $\mathcal{D}$  and  $\mathcal{D}^*$  appears to be very different.

On the other hand, we have also found a *pathological* example in which the  $\mathcal{D}$  and  $\mathcal{D}^*$  charge groups do not coincide.<sup>3</sup> Indeed, for  $SU(4)/\mathbb{Z}_4$  at  $k=4$ , the  $\mathcal{D}$  charge group

<sup>3</sup>MRG thanks Pedro Schwaller for helping him check this carefully.

equals  $\mathbb{Z}_4 \oplus 2 \cdot \mathbb{Z}_2$ , while the  $\mathcal{D}^*$  charge group is  $2 \cdot \mathbb{Z}_4$ . Again this seems to show that the pathological case is structurally rather different. It would be interesting to see whether the other modular invariant [27] that can be defined in this case behaves better.

Nevertheless the striking agreement for the non-pathological cases suggests that there is a more conceptual way in which this can be understood. In the following we describe some preliminary steps towards this goal.

### 5.1 The intertwiner

It is more natural to think about the correspondence between the two charge groups in the context where we consider, for the modular invariant (1.11) say, the untwisted and the twisted D-branes. The open strings between the untwisted D-branes are then characterised by the NIM-rep  $\mathcal{D}$ , while those between the twisted D-branes are described by  $\mathcal{D}^*$ . If we consider both sets of branes simultaneously, we can therefore combine the two NIM-reps into

$$\mathcal{N}_\lambda^{\text{full}} = \begin{pmatrix} \mathcal{N}(\mathcal{D})_\lambda & 0 \\ 0 & \mathcal{N}(\mathcal{D}^*)_\lambda \end{pmatrix}. \tag{5.1}$$

However, we can now also consider the open strings between an untwisted and a twisted D-brane. They will transform in a twisted representation of the affine algebra, and thus we have in addition the off-diagonal matrix [20]

$$\mathcal{N}_{\hat{a}}^{\text{full}} = \begin{pmatrix} 0 & \rho_{\hat{a}} \\ \rho_{\hat{a}}^t & 0 \end{pmatrix}. \tag{5.2}$$

Given the explicit formulae for the  $\psi$ -matrix of the boundary states, it is in principle straightforward to calculate these matrix elements. However, it would be useful to have explicit formulae for these matrix elements in terms of suitable fusion rule coefficients. For example, if  $n$  is odd and  $\text{gcd}(d, k) = 1$ , there are no fixed points, and one simply finds that

$$\rho_{\hat{a},[\nu]}^{(\hat{b},i)} = \mathcal{N}(\mathcal{A}^*)_{\nu \hat{a} \hat{b}}, \tag{5.3}$$

*i.e.* that  $\rho$  agrees with the twisted fusion rules. It is not difficult to check that this definition is well-defined (*i.e.* independent of which representative  $\nu$  in the orbit  $[\nu]$  is chosen). Furthermore, the resulting full NIM-rep forms indeed a representation of the full fusion ring that includes untwisted and twisted representations [20] (see also [31]). This last property implies, in particular, that  $\rho$  defines an intertwiner between the two NIM-reps, *i.e.*

$$\sum_{(\hat{b},r)} \rho_{\hat{a},([\nu],i)}^{(\hat{b},r)} \mathcal{N}(\mathcal{D}^*)_{\lambda,(\hat{b},r)}^{(\hat{c},s)} = \sum_{([\mu],j)} \mathcal{N}(\mathcal{D})_{\lambda,([\nu],i)}^{([\mu],j)} \rho_{\hat{a},([\mu],j)}^{(\hat{c},s)}. \tag{5.4}$$

Here, as in the following, we are labelling the boundary states of the  $\mathcal{D}^*$  NIM-rep by  $(\hat{b}, r)$ ; this is, as we have seen, appropriate as long as the theory is not pathological. The boundary states of the  $\mathcal{D}$  NIM-rep are labelled by  $([\nu], i)$ .

This off-diagonal matrix  $\rho$  defines therefore a map from the untwisted to the twisted D-branes that intertwines the NIM-rep action. Furthermore, there is a canonical ‘smallest’

twisted representation, namely the representation  $\hat{0}$  that has a one-dimensional highest-weight space. It is therefore natural to guess that the off-diagonal matrix associated to  $\hat{0}$  maps the minimal charged D-brane of  $\mathcal{D}$  to the minimal charged D-brane of  $\mathcal{D}^*$ , and that this can ‘explain’ the equivalence of the charge groups. In fact, this is how the situation works for the case of the untwisted and twisted NIM-reps  $\mathcal{A}$  and  $\mathcal{A}^*$  of the simply connected group. In that case  $\rho$  can naturally be identified with the twisted fusion rules themselves, and the minimal solutions are indeed related as

$$\dim(\lambda) = \sum_{\hat{a}} \mathcal{N}(\mathcal{A}^*)_{\lambda \hat{0}}^{\hat{a}} \dim(\hat{a}) \pmod{M} . \tag{5.5}$$

One may hope to be able to use this idea to *prove* the a priori non-trivial fact that the two charge groups of  $\mathcal{A}$  and  $\mathcal{A}^*$  must be the same, but we have not yet succeeded in doing so. The main problem is that we do not understand how to invert this relation, *i.e.* how to express the charges of the twisted branes  $\dim(\hat{a})$  in terms of those of the the untwisted branes  $\dim(\lambda)$ .

The analogue of (5.5) can always be defined: suppose that we have a solution  $q_{\mathcal{D}^*}(\hat{a}, i)$  of the  $\mathcal{D}^*$  charge equations mod some  $M$ . Then we can define a solution of the  $\mathcal{D}$  charge equations by

$$q_{\mathcal{D}}([\nu], l) \equiv \sum_{(\hat{a}, i)} \rho_{\hat{0},([\nu], l)}^{(\hat{a}, i)} q_{\mathcal{D}^*}(\hat{a}, i) . \tag{5.6}$$

It is not difficult to see that the  $q_{\mathcal{D}}([\nu], j)$  then satisfy the  $\mathcal{D}$  charge equations mod the same  $M$ . Indeed, we have

$$\begin{aligned} \dim(\lambda) q_{\mathcal{D}}([\nu], l) &= \sum_{(\hat{a}, i)} \rho_{\hat{0},([\nu], l)}^{(\hat{a}, i)} \dim(\lambda) q_{\mathcal{D}^*}(\hat{a}, i) \\ &= \sum_{(\hat{a}, i)} \sum_{(\hat{b}, j)} \rho_{\hat{0},([\nu], l)}^{(\hat{a}, i)} \mathcal{N}(\mathcal{D}^*)_{\lambda, (\hat{a}, i)}^{(\hat{b}, j)} q_{\mathcal{D}^*}(\hat{b}, j) \pmod{M} \\ &= \sum_{([\mu], m)} \sum_{(\hat{b}, j)} \rho_{\hat{0},([\mu], m)}^{(\hat{b}, j)} \mathcal{N}(\mathcal{D})_{\lambda, ([\nu], l)}^{([\mu], m)} q_{\mathcal{D}^*}(\hat{b}, j) \\ &= \sum_{([\mu], m)} \mathcal{N}(\mathcal{D})_{\lambda, ([\nu], l)}^{([\mu], m)} q_{\mathcal{D}}([\mu], m) , \end{aligned} \tag{5.7}$$

where we have used the NIM-rep property (5.4) in the third line.

## 5.2 Symmetries and resolutions

While (5.6) associates to any solution of the  $\mathcal{D}^*$  charge equations a solution of the  $\mathcal{D}$  charge equations, it is a priori not clear whether all different solutions of  $\mathcal{D}$  can be obtained in this manner. In fact, it is not difficult to find a counterexample to this: already for  $SU(3)/\mathbb{Z}_3$  at level  $k = 3$  the above map does not produce all different solutions of the  $\mathcal{D}$  charge equations. However, it is also not difficult to understand the reason for this: at  $k = 3$  the  $SU(3)/\mathbb{Z}_3$  has additional symmetries, and these can be used to ‘resolve’ the intertwiner  $\rho_{\hat{0}}$  further. To understand how this can be done, we observe that one can associate conserved

charges to the boundary states.<sup>4</sup> For the case of the  $\mathcal{D}^*$  boundary states, the construction is easy: we simply define  $Q(\hat{b}, r) = r$ . This gives indeed a conserved charge since

$$\mathcal{N}(\mathcal{D}^*)_{\lambda,(\hat{b},r)}^{(\hat{c},s)} \neq 0 \implies t(\lambda) + Q(\hat{b}, r) = Q(\hat{c}, s) \pmod{d} . \quad (5.8)$$

What are the possible conserved charges for the  $\mathcal{D}$  boundary states? In general, the charge  $Q([\nu], i)$  cannot depend on  $i$  since the  $\mathcal{D}$  NIM-rep is independent of  $i$  if  $\nu$  is not a fixed point. The only conserved charges of the fusion ring of  $\mathfrak{su}(n)$  are multiples of the  $n$ -ality, but for the quotient theory in question we need them to be independent of the representatives in the  $J^{d'}$  orbits. One easily checks that

$$t\left(J^{d'} \nu\right) = d'k + t(\nu) . \quad (5.9)$$

This means that we can define  $Q([\nu], i) = t(\nu)$ , but that the resulting charge is only defined mod  $R' = \text{gcd}(n, nk/d)$  (since  $n$ -ality is only defined mod  $n$ ). Since the charges for the  $\mathcal{D}^*$  boundary states are only defined mod  $d$ , we have combined charges defined mod  $R = \text{gcd}(d, R') = \text{gcd}(d, d'k)$ .

The idea is now that we can ‘resolve’ each  $\rho$ -intertwiner (and in particular the one associated to  $\hat{0}$ ) into  $R$  intertwiners. Explicitly we define

$$\rho_{[\hat{0},x],[\nu],m}^{(\hat{a},i)} \equiv \rho_{\hat{0},([\nu],m)}^{(\hat{a},i)} \delta^{(R)}(Q(\hat{a}, i) - Q([\nu], m) - x) . \quad (5.10)$$

By construction, we have

$$\rho_{\hat{0}} = \sum_{x=1}^R \rho_{[\hat{0},x]} . \quad (5.11)$$

Thus the  $\rho_{[\hat{0},x]}$  define a ‘resolution’ of the original intertwiner  $\rho_{\hat{0}}$ . Each  $\rho_{[\hat{0},x]}$  is in fact separately an intertwiner: this follows directly from the fact that  $\rho_{\hat{0}}$  is, together with the property of the two charges  $Q(\hat{a}, i)$  and  $Q([\nu], m)$  to be conserved.

In particular, each resolved intertwiner  $\rho_{[\hat{0},x]}$  can therefore also play the role of  $\rho_{\hat{0}}$  in (5.6). In fact, for  $\text{SU}(3)/\mathbb{Z}_3$  at  $k = 3$  we have  $R = 3$ , and we can therefore resolve  $\rho_{\hat{0}}$  into 3 intertwiners. One then easily checks that these resolved intertwiners now do account for all the different charge solutions for the  $\mathcal{D}$  NIM-rep in terms of those of the  $\mathcal{D}^*$  NIM-rep. We suspect that this will always hold, *i.e.* that (5.6) with  $\rho_{\hat{0}}$  replaced by the resolved intertwiners  $\rho_{[\hat{0},x]}$  always accounts for all charge solutions of  $\mathcal{D}$  in terms of those of  $\mathcal{D}^*$ . However, we have so far not been able to prove this for the general (non-pathological) case.

## 6. Conclusions

In this paper we have studied the twisted branes of the WZW model corresponding to the non-simply connected group manifold  $\text{SU}(n)/\mathbb{Z}_d$ . In particular, we have found a very explicit formula for the multiplicities (NIM-rep) with which the different affine representations appear in the relative open strings between these branes. At least in the non-pathological

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<sup>4</sup>These charges should *not* be confused with the D-brane charges we have discussed before!

case  $(n(n+1)/d$  even) this NIM-rep is remarkably simple: if  $n$  is odd it is given by (2.2), while for  $n$  even we have instead (3.3) subject to the constraint (3.5).

Given these simple formulae, we have calculated the charge group that is generated by these twisted D-branes. For  $n$  odd we have proven that the result is given by (2.16). We have also given some convincing evidence that the same result holds for even  $n$ .

We have also made progress towards the description of the NIM-rep for the untwisted branes. In particular, using fixed point factorisation techniques, we have found simple formulae for the NIM-rep coefficients for the fundamental weights (4.3) and (4.4). Using these explicit expressions we have managed to perform a more careful analysis of the corresponding charge group than was possible in [26]. While we have not succeeded in determining it in general, we have analysed it for some examples (in particular  $SU(9)/\mathbb{Z}_9$  at level  $k = 18$ , as well as an infinite class of the form  $SU(n)/\mathbb{Z}_p$  for all  $k$ ), and we have given good evidence that it is again given by (2.16).

Our results thus provide convincing support for the assertion that the charge groups for the untwisted and twisted D-branes agree also for the non-simply connected WZW models  $SU(n)/\mathbb{Z}_d$ , provided that the theory is not pathological, *i.e.* that  $n(n+1)/d$  is even. If we turn the argument around and *assume* that the two charge groups are equal in the non-pathological cases, it would be easy to prove that both are given by (2.16). In fact, the only gap in our argument for the  $\mathcal{D}$ -charge group concerns the unresolved solution that gives rise to the summand  $\mathbb{Z}_M$ ; the existence of this solution is immediate in the  $\mathcal{D}^*$  case. Conversely, the gap in the argument for  $\mathcal{D}^*$  concerns the different solutions with  $q(\widehat{0}, i) \neq q(\widehat{0}, j)$  that correspond to the resolved solutions in the  $\mathcal{D}$  case. Given our improved understanding of the  $\mathcal{D}$ -NIM-rep these are now under good control. It would therefore be very interesting if one could establish the equivalence of these two charge groups abstractly; first steps in this direction were described in section 5.

It would also be very interesting to calculate the relevant K-theory groups using a geometrical approach, and compare the results with the predictions of our conformal field theory analysis. Since the charge groups that were derived above have a very rich structure, this would be a very convincing consistency check of the whole approach.

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## A. Proof of Proposition 2

In this appendix we give the proof of Proposition 2(a). First note that  $\bar{D}_e$  must divide  $n^e$ . To see this, expand the tensor product  $\Lambda_1 \otimes \cdots \otimes \Lambda_1$  out into a sum of irreducible  $\lambda$ , and



hence write  $\dim(\Lambda_1)^e = n^e$  as a sum (with multiplicities) of certain  $\dim(\lambda)$ , where each  $t(\lambda) = e \pmod n$ . Since  $\bar{D}_e$  divides each such  $\dim(\lambda)$ , it must divide their sum.

Let  $p^\nu \parallel n$ . Then repeatedly applying Proposition 1(b), we get for any  $j$  that

$$\dim(\Lambda_{p^j \ell}) = \binom{n/p^j}{\ell} \pmod{p^{\nu-j+1}}. \tag{A.1}$$

Thus if  $p$  does not divide  $d$ , then  $(\dim(\Lambda_{p^\nu}))^e = (n/p^\nu)^e \pmod p$  will be coprime to  $p$ , but as in the previous paragraph must be divisible by  $\bar{D}_e$ . This tells us that  $\bar{D}_e$  must in fact divide  $d^\infty$ .

Now suppose  $p$  divides  $d$ . Taking the gcd of eq. (A.1), over all  $0 < \ell < n/p^j$  coprime to  $p$ , we get  $p^{\nu-j}$  (by Proposition 1(a) applied to  $\text{su}(n/p^j)$ ). So  $p^{\nu-j}$  divides each  $\dim(\Lambda_{p^j \ell})$ , when  $\ell$  is coprime to  $p$ .

Now, consider any weight  $\lambda$  with  $\text{gcd}(t(\lambda), \bar{D}) = e$ . The  $\text{su}(n)$ -character  $\text{ch}_\lambda$  will be a polynomial, with coefficients in  $\mathbb{Z}$ , in the fundamental weights  $\text{ch}_{\Lambda_\ell}$ , so  $\dim(\lambda)$  will equal that same polynomial, evaluated at  $\dim(\Lambda_\ell)$ . This polynomial will be homogeneous (in the obvious weighted sense) of degree  $t(\lambda) \pmod n$ . Let  $p^\epsilon \parallel e$ , for some prime  $p$ . We first want to show that  $p^{\nu-\epsilon}$  divides each term in that polynomial, *i.e.* that it divides each product  $\prod_i \dim(\Lambda_{\ell_i})$  of total weighted degree  $t \equiv \sum_i \ell_i$  satisfying  $\text{gcd}(t, \bar{D}) = e$ .

But this is clear. The given term must contain some  $\dim(\Lambda_{\ell p^h})$ , where  $\ell$  is coprime to  $p$  and  $0 \leq h \leq \epsilon$ . So  $p^{\nu-h}$  (and hence  $p^{\nu-\epsilon}$ ) will divide  $\dim(\Lambda_{\ell p^h})$  and hence the whole term.<sup>5</sup> Repeating for all  $p$ , we find that  $\bar{D}/e = \prod_{p|d} p^{\nu-\epsilon}$  will divide  $\bar{D}_e$ .

All that remains is to show the other direction, *i.e.* for each prime  $p$  to find a weight  $\lambda$  with  $\text{gcd}(t(\lambda), \bar{D}) = e$ , and with  $p^{\nu-\epsilon} \parallel \dim(\lambda)$ . To do this, consider the tensor product  $\Lambda_{p^\epsilon} \otimes \Lambda_{p^\nu t}$ , where we take  $0 \leq t < n/p^\nu$  so that  $p^\epsilon + p^\nu t = e \pmod n$ . From eq. (A.1),  $p^{\nu-\epsilon}$  exactly divides  $\dim(\Lambda_{p^\epsilon})$ , while  $p$  is coprime to  $\Lambda_{p^\nu t}$ , and so  $p^{\nu-\epsilon}$  exactly divides that product of dimensions. Expanding that product out into a sum of dimensions,  $p^{\nu-\epsilon}$  will divide each dimension separately in that sum, and so it must divide exactly at least one dimension in that sum. That is the desired  $\dim(\lambda)$ . This argument breaks down when  $p^\epsilon = n$ , but in this case  $\lambda = \Lambda_1 + \Lambda_{n-1}$ , with dimension  $n^2 - 1$  coprime to  $p$ , works. **QED.**

## B. Fixed point factorisation and NIM-reps

Consider any non-pathological  $\text{SU}(n)/\mathbb{Z}_d$  at level  $k$  ( $n$  can be even or odd). Let  $f = \text{gcd}(d, k)$  and write  $d' = n/d$  as usual.

Let  $[\varphi, i], [\varphi', i']$  be boundary states with orders  $o(\varphi) = g, o(\varphi') = g'$ , respectively. Let  $(\psi, j)$  be an exponent with order  $o(\psi) = h$ . We will let superscript  $(\delta)$  denote quantities associated with  $\widehat{\text{su}}(n/\delta)$  at level  $k/\delta$ . We let  $0$  and  $J$  denote the vacuum and simple current generator in any  $\widehat{\text{su}}(n/\delta)$  level  $k/\delta$ .

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<sup>5</sup>We are being a little sloppy here: when  $\epsilon = \nu$  it is possible for all  $h$  to exceed  $\epsilon$ , but in that boundary case we only have to prove that  $p^{\nu-\epsilon} = 1$  divides the dimensions, which is trivial.

Assume we are in **Case A** (this includes all non-pathological, but also some pathological, cases). Then the formula of [32, 18] is

$$\psi_{[\varphi,i],(\psi,j)} = \frac{\sqrt{d}}{g\sqrt{h}} \sum_{\delta|\gcd(g,h)} \xi_\delta s(\delta, i-j) S_{\varphi^\delta \psi^\delta}^{(\delta)}, \quad (\text{B.1})$$

where  $\xi_\delta$  is some irrelevant root of unity depending on  $\delta, n, k, d$ , and

$$s(a, b) = \sum_{\ell \in \mathbb{Z}_a^\times} e^{2\pi i \ell b/a} \quad (\text{B.2})$$

(that is, the sum is over all  $1 \leq \ell \leq a$  coprime to  $a$ ). In fact, this quantity  $s(a, b)$  is called a Ramanujan sum, and is easily seen to equal

$$s(a, b) = \sum_{d|\gcd(a,b)} \mu(n/d) d, \quad (\text{B.3})$$

where  $\mu(c)$  is the *Möbius function*, which equals 0 unless  $c$  is the product of  $s \geq 0$  distinct primes, in which case  $\mu(c) = (-1)^s$ . We will need one other property of  $s(a, b)$ : provided  $a, b$  divide  $h$ ,

$$\sum_{j=1}^h s(a, i-j) s(b, j-i') = h s(a, i-i') \delta_{a,b}. \quad (\text{B.4})$$

This identity follows directly from (B.2) and the calculation

$$\begin{aligned} & \sum_{j=1}^h \sum_{\ell \in \mathbb{Z}_a^\times} \sum_{m \in \mathbb{Z}_b^\times} e^{2\pi i \ell(i-j)/a} e^{2\pi i m(j-i')/b} \\ &= \sum_{\ell \in \mathbb{Z}_a^\times} \sum_{m \in \mathbb{Z}_b^\times} \exp[2\pi i (\frac{\ell i}{a} - \frac{m i'}{b})] \sum_{j=1}^h \exp \left[ 2\pi i \frac{j}{h} (\frac{mh}{b} - \frac{lh}{a}) \right] \\ &= \sum_{\ell \in \mathbb{Z}_a^\times} \sum_{m \in \mathbb{Z}_b^\times} \exp[2\pi i (\frac{\ell i}{a} - \frac{m i'}{b})] \delta_{\ell, m} \delta_{a,b} = h s(a, i-i') \delta_{a,b}. \end{aligned} \quad (\text{B.5})$$

Plugging everything into (1.12) gives a mess, even at the fundamental weights:

$$\begin{aligned} \mathcal{N}_{\Lambda_m, [\varphi, i]}^{[\varphi', i']} &= \frac{d}{gg'} \sum_{h|f} \sum_{\substack{t(\psi)=d^0 \\ o(\psi)=h}} \frac{1}{h} \sum_{j=1}^h \left( \sum_{\delta|\gcd(g,h)} \xi_\delta s(\delta, i-j) S_{\varphi^\delta \psi^\delta}^{(\delta)} \right) \frac{S_{\Lambda_m \psi}}{S_{0\psi}} \\ & \quad \left( \sum_{\delta'|\gcd(g',h)} \xi_{\delta'}^* s(\delta', j-i') S_{\varphi'^{\delta'} \psi^{\delta'}}^{(\delta')*} \right). \end{aligned} \quad (\text{B.6})$$

The key observation is that, because of (B.4), the crossterms in (B.6) (that is the terms with  $\delta \neq \delta'$ ) vanish, and what we obtain is

$$\mathcal{N}_{\Lambda_m, [\varphi, i]}^{[\varphi', i']} = \frac{d}{gg'} \sum_{h|f} \sum_{\substack{t(\psi)=d^0 \\ o(\psi)=h}} \sum_{\delta|\gcd(g,h,g',m)} s(\delta, i-i') S_{\varphi^\delta \psi^\delta}^{(\delta)} \frac{S_{\Lambda_m/\delta \psi^\delta}^{(\delta)}}{S_{0\delta \psi^\delta}^{(\delta)}} S_{\varphi'^{\delta} \psi^\delta}^{(\delta)}, \quad (\text{B.7})$$

where we have used fixed point factorisation. The restriction to  $t(\psi) = 0 \pmod{d}$  is easily obtained by a sum over simple currents:  $\frac{1}{d} \sum_{j=1}^d$ . However, we can write  $\frac{d}{g''} = a\frac{d}{g} + b\frac{d}{g'}$ , where  $g'' = \text{lcm}(g, g')$ , so by (1.8) it suffices to take the sum  $\frac{g''}{d} \sum_{j=1}^{d/g''}$ . We obtain

$$\mathcal{N}_{\Lambda_m, [\varphi, i]}^{[\varphi', i']} = \frac{g''}{gg'} \sum_{\delta | \text{gcd}(g, g', m)} s(\delta, i - i') \sum_{j=1}^{d/g''} N_{J^{d'j} \Lambda_{m/\delta} \varphi^\delta}^{(\delta) \varphi'^\delta} . \tag{B.8}$$

Again,  $N^{(\delta)}$  denotes fusion coefficients for  $\widehat{\text{su}}(n/\delta)$  level  $k/\delta$ . By substituting in (B.3) and rearranging the sum, we can rewrite (B.8) in a more friendly form:

$$\mathcal{N}_{\Lambda_m, [\varphi, i]}^{[\varphi', i']} = \frac{g''}{gg'} \sum_{j=1}^{d/g''} \sum_{a|\Delta} a \delta_{ii'}^{(a)} \sum_{\delta, a|\delta|\Delta} \mu(\delta/a) N_{J^{d'j} \Lambda_{m/\delta} \varphi^\delta}^{(\delta) \varphi'^\delta} , \tag{B.9}$$

where  $\Delta = \text{gcd}(g, g')$ , and where  $\delta_{ij}^{(x)}$  equals 0 unless  $x$  divides  $i - j$ , when it equals 1. The second sum is over all multiples  $\delta$  of  $a$ , which divide  $\Delta$ . If some  $\delta$  does not divide  $m$  then we interpret the entire right-side of (B.9) as equalling 0. That the coefficient  $\mathcal{N}_{\Lambda_m, [\varphi, i]}^{[\varphi', i']}$  must vanish if  $\Delta$  (or any other  $\delta$ ) does not divide  $m$ , is clear from (4.2) and positivity.

A special case of (B.9) will be very useful. If  $\Delta$  equals a prime  $p$ , and  $d = g''$  and  $p$  divides  $m$ , then (B.9) simplifies to

$$\mathcal{N}_{\Lambda_m, [\varphi, i]}^{[\varphi', i']} = \frac{1}{p} \left( N_{\Lambda_m \varphi}^{\varphi'} - N_{\Lambda_{m/p} \varphi^p}^{(p) \varphi'^p} + p \delta_{ii'}^{(p)} N_{\Lambda_{m/p} \varphi^p}^{(p) \varphi'^p} \right) . \tag{B.10}$$

But these coefficients must be integral, and thus  $N_{\Lambda_m \varphi}^{\varphi'}$  and  $N_{\Lambda_{m/p} \varphi^p}^{(p) \varphi'^p}$  should be congruent  $\pmod{p}$ . But by the Pieri formula, tensor product coefficients (hence fusion coefficients) involving fundamental weights will always be 0 or 1. Thus we obtain

$$N_{\Lambda_m \varphi}^{\varphi'} = N_{\Lambda_{m/p} \varphi^p}^{(p) \varphi'^p} , \tag{B.11}$$

which must hold for any multiples  $n, k, m$  of  $p$ , and weights  $\varphi, \varphi'$  fixed by  $J^{n/p}$ . Using this, (B.9) simplifies when  $\Delta = p$ , but as we will see shortly much more is true.

Even formula (B.9) is surprisingly simple. But we can reduce it much more, by induction on  $d$ . Suppose for all  $n, k$ , we get the following formula — the main result of this appendix — valid for any possible  $d < D$  (of course  $d$  must divide  $n$ ):

$$\mathcal{N}_{\Lambda_m, [\varphi, i]}^{[\varphi', i']} = \sum_{j=1}^{d/g''} \delta_{ii'}^{(\Delta)} N_{J^{d'j} \Lambda_{m/\Delta} \varphi^\Delta}^{(\Delta) \varphi'^\Delta} \tag{B.12}$$

$$\mathcal{N}_{J0, [\varphi, i]}^{[\varphi', i']} = \delta_{[\varphi'], [J\varphi]} \delta_{i, i'} , \tag{B.13}$$

where again we interpret the right-side of (B.12), (B.13) as vanishing if  $\Delta$  does not divide  $m$ . We want to show that the induction hypothesis (B.12) will also hold for  $d = D$ . This would then imply that (B.12) always holds.

Certainly the induction hypothesis holds for  $D = 2$ . In fact from results in [26] (or using (B.11) in (B.9)) we know it holds for  $D = 4$  as well. So take  $d = D$ , and any multiple

$n$  of  $d$ . We may assume that  $\Delta = \gcd(g, g')$  divides  $m$ . Consider any prime  $p$  dividing  $\Delta$  — say it divides  $\Delta$  exactly  $\alpha$  times. Write  $n_0 = n/p^\alpha$ ,  $d_0 = d/p^\alpha$ ,  $k_0 = k/p^\alpha$ ,  $\Delta_0 = \Delta/p^\alpha$ . Note that  $\varphi^{p^\alpha}$  and  $\varphi'^{p^\alpha}$  have orders  $g_0 = g/p^\alpha$  and  $g'_0 = g'/p^\alpha$ . We want to reduce (B.12), (B.13) for  $n, k, d$ , to (B.12), and (B.13) for  $n_0, k_0, d_0$ .

Assume first that  $\alpha \geq 2$ . Then by virtue of the Möbius function, we can write

$$\sum_{a|\Delta} a \delta_{ii'}^{(a)} \sum_{\delta, a|\delta|\Delta} \mu(\delta/a) N_{\lambda(\delta)}^{(\delta)} \varphi^\delta \varphi'^{\delta} = \sum_{b=0}^{\alpha} p^b \delta_{ii'}^{(p^b)} \sum_{a_0|\Delta_0} a_0 \delta_{ii'}^{(a_0)} \sum_{\delta_0, a_0|\delta_0|\Delta_0} \mu(\delta_0/a_0) \left( N_{\lambda(p^b \delta_0)}^{(p^b \delta_0)} \varphi^{p^b \delta_0} \varphi'^{p^b \delta_0} - N_{\lambda(p^{b+1} \delta_0)}^{(p^{b+1} \delta_0)} \varphi^{p^{b+1} \delta_0} \varphi'^{p^{b+1} \delta_0} \right), \quad (\text{B.14})$$

where we drop all terms with  $p^{\alpha+1}$ . The weight we call  $\lambda(\delta)$  here is  $J^{d'} \Lambda_{m/\delta}$ , but it could be anything. If instead  $\alpha = 1$ , then again (B.14) must hold.

The term in the brackets of (B.14) equals 0, by virtue of (B.11). Thus everything on the right-side of (B.14) vanishes, except for the  $b = \alpha$  terms, and we get

$$\begin{aligned} \sum_{a|\Delta} a \delta_{ii'}^{(a)} \sum_{\delta, a|\delta|\Delta} \mu(\delta/a) N_{\lambda(\delta)}^{(\delta)} \varphi^\delta \varphi'^{\delta} \\ = p^\alpha \delta_{ii'}^{(p^\alpha)} \sum_{a_0|\Delta_0} a_0 \delta_{ii'}^{(a_0)} \sum_{\delta_0, a_0|\delta_0|\Delta_0} \mu(\delta_0/a_0) N_{\lambda(p^\alpha \delta_0)}^{(p^\alpha \delta_0)} \varphi^{p^\alpha \delta_0} \varphi'^{p^\alpha \delta_0}. \end{aligned} \quad (\text{B.15})$$

Substituting this into (B.9), we obtain

$$\begin{aligned} \mathcal{N}_{\Lambda_m, [\varphi, i]}^{[\varphi', i']} &= \frac{g_0''}{p^\alpha g_0 g_0'} \sum_{j=1}^{d_0/g_0''} p^\alpha \delta_{ii'}^{(p^\alpha)} \sum_{a_0|\Delta_0} a_0 \delta_{ii'}^{(a_0)} \sum_{\delta_0, a_0|\delta_0|\Delta_0} \mu(\delta_0/a_0) N_{J^{d'} \Lambda_{\frac{m}{p^\alpha \delta_0}}}^{(p^\alpha \delta_0)} \varphi^{p^\alpha \delta_0} \varphi'^{p^\alpha \delta_0} \\ &= \mathcal{N}_{\Lambda_{\frac{m}{p^\alpha}}, [\varphi^{p^\alpha}, i \pmod{p^\alpha}]}^{[\varphi'^{p^\alpha}, i' \pmod{p^\alpha}]}. \end{aligned} \quad (\text{B.16})$$

By the induction hypothesis, this gives us (B.12), and we are done.

### C. Proofs for $\mathcal{D}$ charges

Consider any  $SU(n)/\mathbb{Z}_d$  level  $k$  in Case A. Let  $q[\lambda, i]$  be a twisted charge solution, taken mod  $M$ , to  $\mathcal{D}(SU(n)/\mathbb{Z}_d)$ , *i.e.*  $q[\lambda, i] = 0$  unless  $\lambda$  is a fixed point of some non-trivial power of  $J^{d'}$ . Let  $\phi$  be such a fixed point, of order  $o(\phi)$ . Then considering the charge equation for  $0 = \dim(\phi) q[0]$ , we find that full sums are 0:

$$\sum_{h=1}^{o(\phi)} q[\phi, h] = 0 \pmod{M}. \quad (\text{C.1})$$

Now, for any integer  $m$  coprime to  $o(\phi)$ , the charge equation for  $\dim(\Lambda_m) q[\phi, i]$  will be full sums, by (4.1), and so will be 0 mod  $M$ . By Proposition 2(b), the gcd of those  $\dim(\Lambda_m)$  will be  $\gcd(o(\phi)^\infty, n)$ . Since in addition  $Mq[\phi, i] = 0 \pmod{M}$ , we have that  $Dq[\phi, i] = 0$

(mod  $M$ ), where  $D = \gcd(M, (o(\phi)^\infty, n))$ . This tells us that the twisted charge group is a subgroup of  $\mathbb{Z}_n^\infty$  and can be built up out of the primes dividing  $d$ . (This argument requires  $o(\Lambda_m) = 1$ . We will run into problems here only if  $k = 2$ ,  $m = n/2$ , in which case  $o(\phi) = 2$  and  $m$  must be odd, but this would be Case B, which is excluded.)

Now choose any prime  $p$  dividing  $d$ , and put  $p^\nu \parallel n$ ,  $p^\delta \parallel d$ . Let  $p^\ell$  exactly divide  $o(\phi)$ , i.e.  $p^\ell$  is the  $J^{n/p^\nu}$ -order of fixed point  $\phi$ . Write  $p^\gamma = \gcd(M, n, p^\infty)$ . We claim  $i = j \pmod{p^\ell}$  implies  $q[\phi, i] = q[\phi, j] \pmod{p^\gamma}$ .

To see this, let  $Q$  be the product of all primes  $\neq p$  dividing  $o(\phi)$ . By (4.3), we have  $\dim(\Lambda_m) q[\phi, i] = \dim(\Lambda_m) q[\phi, j] \pmod{M}$  whenever  $i = j \pmod{p^\ell}$ . Now run over all  $m$  coprime to  $Q$  and apply Proposition 2(b).

We are now prepared to show that it suffices to analyse prime-power orbifolds of  $SU(n)$ . In particular, given twisted charges  $q[\phi, i]$  for  $\mathcal{D}(SU(n)/\mathbb{Z}_d)$ , define  $q_p[\phi, i]$  (for  $1 \leq i \leq p$ ) to be  $(o(\phi)/p^\ell)q[\phi, i]$ . Then the  $q_p$  will solve the charge equations (mod  $p^\gamma$ ) for  $\mathcal{D}(SU(n)/\mathbb{Z}_{p^\delta})$ . The converse is also true: given twisted charges  $q_p$  for  $\mathcal{D}(SU(n)/\mathbb{Z}_{p^\delta})$ , for each prime dividing  $d$ , you can get a twisted solution  $q$  for  $\mathcal{D}(SU(n)/\mathbb{Z}_d)$ , defined by the same formula. This establishes a bijection between the twisted charge group of  $\mathcal{D}(SU(n)/\mathbb{Z}_d)$  and the direct product over  $p$  of the full  $\mathcal{D}$ -charge groups of  $SU(n)/\mathbb{Z}_{p^\delta}$ . The proof is straightforward, using (4.3), although the book-keeping is messy.

Similar arguments explain how the *untwisted*  $\mathcal{D}$ -charges of  $SU(n)/\mathbb{Z}_d$  are related to the  $\mathcal{D}$ -charges for  $SU(n)/\mathbb{Z}_{p^\delta}$ . The relation between twisted  $\mathcal{D}$ -charges for  $SU(n)/\mathbb{Z}_{p^\delta}$  and those of  $SU(n)/\mathbb{Z}_{p^{\delta-1}}$ , follows directly from (4.3).

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